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ROYAL SIGNALS & RADAR ESTABLISHMENT

POLARIMETRIC PRINCIPLES AND TECHNIQUES

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Author: S R Cloude

Date: October 1983

SUMMARY

This memorandum outlines the mathematical formulation of a polarimetric theory for radar scattering. The emphasis is placed on physical interpretation of some fundamental results from the theory of nonsingular linear transformations and the general scattering problem treated as a geometrical transformation on the Poincaré sphere.

An introduction to second order statistical effects in polarimetric scattering is also provided via the coherency matrix and Stokes vector. The matrix governing transformation of these second order parameters is related to the elements of the coherent scattering matrix.

This memorandum was derived from a set of lectures given by the author at Birmingham University in July 1983.

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POLARIMETRIC PRINCIPLES AND TECHNIQUES

S R Cloude

LIST OF CONTENTS

INTRODUCTION

When an object of arbitrary size, shape and material scatters an incident electromagnetic wave, measurement of the parameters of the scattered field may be used to yield information about that object. One very important parameter of the wave which has not been fully exploited in the past is its polarisation. On the whole radars have measured amplitude, phase and doppler shift of targets but have remained fixed polarisation devices and thus have not obtained the maximum available information, even for a fixed frequency and aspect. The capability of obtaining complete polarimetric information is therefore an attractive feature, and how this information may be gathered and best used is the subject of these lectures.

Although polarimetric techniques have found widespread use in optical instrumentation and materials analysis for many years, the relevance of polarimetric phenomena to microwave scattering has remained primarily a topic of theoretical interest only. This is unfortunate since at longer wavelengths polarisation is a more significant parameter in scattering than it is in optics and as such promises to yield a considerable amount of target information. Since polarisation may be described as the spatially directive quality of a wave then knowledge of the scattering characteristics of an object might be expected to yield information relating to target symmetry, geometry and material properties. As a simple example consider rain clutter suppression radars which transmit and receive like sense circular polarisation on the basis that this will cancel the backscatter from spherically symmetrical raindrops. It will be shown in the course of these lectures how more subtle indicators of target symmetry may be derived from knowledge of the polarisation transforming properties of the target.

Recently there is a great deal of interest in optimising radar system performance and in target identification studies both of which require a more exact target descriptor than the conventional fixed polarisation radar cross section(1). One way of providing this is to describe the target by a polarisation scattering matrix of finite dimensions with manipulation of this matrix in the radar signal processing yielding target information. Essentially this means that the transmit/receive antenna assembly no longer acts as a polarisation filter but collects full polarimetric information and allows all filtering to be performed in the processing(6). With ongoing advances in the field of digital and analogue signal processing technology this allows for practical exploitation of polarimetric information.

The importance of developing a full polarimetric theory for radar scattering was realised in the late nineteen forties. In particular E.M. Kennaugh demonstrated the importance of polarimetric techniques when, in a series of reports written between 1949 and 1954[2], he introduced the important concept of null or optimal polarisations for radar targets. It was Sinclair[3] in 1948 who was the first to describe the polarisation transformation properties by a 2 x 2 coherent scattering matrix. At about the same time Mueller[4] developed a more general matrix calculus for handling partially polarised waves in optics. This calculus, based on Stokes vectors was applied to radar scattering much later by J.R. Huynen[5] and allows a polarimetric theory to be developed for handling radar scatter from fluctuating targets like chaff or clutter. More recent developments have seen the practical implementation of programmable polarisation filter design and its use for more efficient suppression of rain clutter[7]. Significant theoretical work on incorporating polarimetric techniques with broadband scattering theories has been done by Boerner[8]. Despite these advances the full potential of polarimetric techniques has still not been realised and one can only speculate as to the future. Certainly the next decade will see the emergence of more and more radar systems utilising polarisation for optimised performance.

POLARISATION DEFINITIONS

Before undertaking a full polarimetric description of target scattering it is necessary to consider the various methods used to describe the polarisation of a wave. In this chapter we shall cover three methods: firstly, the orthogonal component decomposition for plane monochromatic waves. The geometrical parameters of the polarisation ellipse will then be related to these plane wave parameters since the former are useful when considering graphical aids to polarisation problems. Finally, the more general Stokes vector formalism shall be developed and related to the other representations through the Poincaré sphere.

In the most general case a propagating electromagnetic wave has six field components, namely three mutually perpendicular electric and magnetic field vectors ($E_x, E_y, E_z, H_x, H_y, H_z$). For most radar applications, however, it is sufficient to consider only the case of plane time harmonic waves travelling in free space so that in general we may write the field as

$$E_x = a_x \cos(\psi + \delta_x) \quad \text{- 1a} \qquad E_{x,y,} = \eta H_{y,x,}$$

η = impedance of free space = 120 π

$$E_y = a_y \cos(\psi + \delta_y) \quad \text{- 1b} \qquad \psi = \omega t - kr$$

δ_x, δ_y = instantaneous phase of $E_{x,y}$ components at $t = 0$

$$E_z = 0$$

ω = angular frequency

$k = \frac{2\pi}{\lambda}$

r = range from source

Here, x,y,z is a right handed set of cartesian co-ordinates with the direction of propagation of the wave in the z direction.

If the plane wave assumption is relaxed to allow partially polarised waves where a_x, a_y, δ_x, δ_y are functions of time then the wave must be specified by its second order statistics or coherency matrix(9)

$$J = \begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_x^* E_y \rangle & \langle E_y E_y^* \rangle \end{bmatrix} \qquad \langle\rangle = \text{time average}$$

Note, significantly, that J is Hermitian and so may be written in the general form

$$J = \begin{bmatrix} a & b-ic \\ b+ic & d \end{bmatrix}$$

Thus four real numbers (a,b,c,d) are required to specify the second order statistics of the wave.

1. ORTHOGONAL COMPONENT DECOMPOSITION

The polarisation of a plane monochromatic wave is defined as the locus of the electric field vector (E_x+E_y) in a plane perpendicular to the direction of propagation. In general the vector direction will change as a function of time, and according to IEEE standard[10], when observed along a line of sight anti-parallel to the direction of propagation is called left handed polarisation if it rotates clockwise and right handed if counter clockwise. If the E vector does not rotate at all, the wave is said to be linearly polarised.

In order to derive an expression for this locus for arbitrary $a_x, a_y, \delta_x, \delta_y$ combine equations 1a and 1b so as to eliminate ψ[9]. When this is done the equation for the locus is

$$\left(\frac{E_x}{a_x}\right)^2 + \left(\frac{E_y}{a_y}\right)^2 - 2\,\frac{E_x E_y}{a_x a_y}\cos\delta_{xy} = \sin^2\delta_{xy} \qquad \delta_{xy} = \delta_y - \delta_x$$

This is the equation of an ellipse for which linear and circular polarisations are special cases. In general, the wave may be represented by the two component column vector

$$\underset{\sim}{E} = \begin{bmatrix} E_x & : & \hat{x} \\ E_y & : & \hat{y} \end{bmatrix} = \mathrm{Re}\begin{bmatrix} a_x & & : & \hat{x} \\ a_y & e^{J\delta_{xy}} & : & \hat{y} \end{bmatrix} e^{J(\tau+\delta_x)}$$

When considering only the state of polarisation of this wave then use will be made of the column vector

$$\hat{p} = \begin{bmatrix} a_x \\ a_y\, e^{J\delta_{xy}} \end{bmatrix}$$

Formally, this vector, each element of which is complex, forms a spinor and much of what follows finds elegant interpretation in terms of spinor algebra[11]. Note that in the above notation $\hat{x}$ and $\hat{y}$ may be any two orthogonal polarisations of which horizontal and vertical and left and right circular are common examples. In general any pair of elliptical polarisations will suffice as long as they satisfy the orthogonality condition

$$\hat{x} \cdot \hat{y}^* = 0$$

Each pair forms a so-called polarisation base written (x,y). Any polarisation $\hat{P}$ may then be represented in terms of a complex combination of these components

$$\hat{P} = A_{\hat{x}} + B_{\hat{y}}$$

$\hat{x},\hat{y}$ - unit orthogonal vectors

A,B - complex coefficients determining $\hat{P}$

Of interest is to consider how to express $\hat{P}$ in terms of an arbitrary base (x',y') when it is known in terms of (x,y). The most general transformation (x',y') from (x,y) has the following normalisation and orthogonality conditions

$$x' \cdot x'^* = 1$$

$$y' \cdot y'^* = 1$$

$$x' \cdot y'^* = 0$$

The transformation will be linear and so may be written

$$x' = P_{\hat{x}} + Q_{\hat{y}}$$

$$y' = R_{\hat{x}} + S_{\hat{y}}$$

P,Q,R,S will be complex

In order to satisfy the above conditions this transformation will have the general form

$$\hat{x}' = \cos\alpha\ e^{i\phi_1}\ \hat{x} + \sin\alpha\ e^{i\phi_2}\ \hat{y} \qquad 0 < \alpha < 90^\circ$$

$$\hat{y}' = -\sin\beta\ e^{i\phi_3}\ \hat{x} + \cos\beta\ e^{i\phi_4}\ \hat{y} \qquad 0 < \beta < 90^\circ$$

From the orthogonality condition

$$-\cos\alpha\ \sin\beta\ e^{i(\phi_1-\phi_3)} + \sin\alpha\ \cos\beta\ e^{i(\phi_2-\phi_4)} = 0$$

This leads to the conditions

$$\phi_1 - \phi_3 = \phi_2 - \phi_4$$

$$\alpha = \beta$$

Further, we can set $\phi_1 = \phi_4 = 0$ without loss of generality and so can write the transformation in matrix notation as

$$\begin{bmatrix} \hat{x}' \\ \hat{y}' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha\ e^{J\delta} \\ -\sin\alpha\ e^{-J\delta} & \cos\alpha \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \qquad \delta = \phi_3$$

This may also be written

$$[T] = \cos\alpha \begin{bmatrix} 1 & \tan\alpha\ e^{J\delta} \\ -\tan\alpha\ e^{-J\delta} & 1 \end{bmatrix}$$

At this point is is useful to introduce the complex polarisation ratio defined as

$$\rho = \frac{B}{A} = |\rho|\ e^{J\phi_\rho}$$

This ratio defines $\hat{P}$ in terms of the ratio of the complex coefficients A and B. From this definition and the general form of [T] it is evident that

$$\rho = \tan\alpha\ e^{J\delta}$$

where $\left|\frac{B}{A}\right| = \tan\alpha \qquad 0 \leq \alpha \leq 90^\circ$

$$\phi_B - \phi_A = \delta \qquad 0 < \delta \leq 360^\circ$$

The geometrical significance of the two angles α, δ will be discussed later in this chapter when considering the Poincaré sphere. Note that in terms of ρ

$$\cos\alpha = (1 + \rho\rho^*)^{-\frac{1}{2}}$$

This then allows us to write $[T]^T$, the transpose of $[T]$, which governs the transformation of coefficients of $(\hat{x}\ \hat{y})$ as

$$[T]^T = \frac{1}{\sqrt{1+\rho\rho^*}} \begin{bmatrix} 1 & -\rho^* \\ \rho & 1 \end{bmatrix}$$

2. GEOMETRICAL PARAMETERS

A convenient method of representation of the polarisation ellipse is in terms of its inclination angle θ ($0 \leq \theta \leq 180^\circ$) and ellipticity angle τ ($-45^\circ \leq \tau \leq 45^\circ$) as defined in Figure 1. By convention, positive values of τ correspond to left hand polarisations and negative values right hand. The amplitude of the wave is defined as in Figure 1.

In order to express an arbitrary state of polarisation $\hat{P}$ in matrix form consider firstly that the x,y axes in Figure 1 are aligned with the major and minor axes of the ellipse respectively. From the definition of τ we can write

$$\underline{E} = \begin{bmatrix} a_x \cos \omega t \\ a_y \cos(\omega t+\delta_{xy}) \end{bmatrix} = \begin{bmatrix} a \cos\tau \cos\omega t \\ -a \sin\tau \sin\omega t \end{bmatrix} = \mathrm{Re} \begin{bmatrix} a \cos\tau \\ Ja \sin\tau \end{bmatrix} e^{J\omega t}$$

In complex notation

$$\hat{P} = a \begin{bmatrix} \cos\tau \\ J \sin\tau \end{bmatrix}$$

For the general case where the major axis of the ellipse makes an angle θ with the axes, this column vector is multiplied by the transformation matrix for rotations in 2-dimensions to yield

$$\hat{P} = a \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\tau \\ J \sin\tau \end{bmatrix} = \hat{P}(a,\theta,\tau)$$

In terms of these geometrical parameters, the orthogonal polarisation to $\hat{P}(a,\theta,\tau)$ is defined as

$$\hat{P}_{\perp} = \hat{P}(a,\ \theta + \frac{\pi}{2},\ -\tau)$$

In other words an ellipse with its major axis rotated through 90° and with the same magnitude of ellipticity but opposite sense (Figure 2). The transformation matrix [T] changing (x,y) into (x',y') comprises two components: one representing a deformation of ellipticity given by angle τ_c and the other a rotation of the major axis by θ_c. Thus

$$[T] = \begin{bmatrix} \cos\theta_c & -\sin\theta_c \\ \sin\theta_c & \cos\theta_c \end{bmatrix} \begin{bmatrix} \cos\tau_c & J\sin\tau_c \\ J\sin\tau_c & \cos\tau_c \end{bmatrix}$$

By expanding this matrix product and comparing terms with [T] written in terms of α, δ, the relationships between the geometrical parameters and the polarisation ratio terms may be derived as[12]

$$\cos2\alpha = \cos2\theta\ \cos2\tau$$

$$\tan\delta = \tan2\tau\ \csc2\theta$$

with the inverse relationships

$$\tan2\theta = \tan2\alpha\ \cos\delta$$

$$\sin2\tau = \sin2\alpha\ \sin\delta$$

STOKES PARAMETERS AND THE COHERENCY MATRIX

The above definitions are adequate when considering coherent waves ie waves where the parameters of the polarisation ellipse are independent of time. More generally it is of interest to consider scattering from fluctuating targets and in these instances the polarisation ellipse will fluctuate as a function of time. In particular $\langle E_x \rangle$ and $\langle E_y \rangle$ may equal zero for uniformly random variations in phase and so in order to describe the wave one must use its coherency matrix[9]. This measures the complex correlation that exists between two orthogonal polarisation vectors over the period of observation. At the two extremes there will be either complete correlation in which case the wave is coherent and elliptically polarised (ep) or zero correlation which implies a randomly polarised wave (rp). In general there will exist some degree of correlation and hence the wave is termed partially polarised (pp).

Previously the coherency matrix was defined as

$$J = \begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_x^* E_y \rangle & \langle E_y E_y^* \rangle \end{bmatrix} = \begin{bmatrix} \langle a_x^2 \rangle & \langle a_x a_y e^{-J\delta_{xy}} \rangle \\ \langle a_x a_y e^{J\delta_{xy}} \rangle & \langle a_y^2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} a & b-ic \\ b+ic & d \end{bmatrix} = \begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix}$$

Note that:

Trace (J) = sum of diagonal elements = $J_{xx} + J_{yy}$ = total power in the wave.

Also, since J_{xx} J_{yy} are non-negative:

$$\det (J) = J_{xx}J_{yy} - J_{xy}J_{yx} \geq 0$$

For rp waves there are no preferred polarisation parameters and so

$$J_{xx} = J_{yy} \text{ and } J_{xy} = J_{yx} = 0$$

$$[J] = I_o \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

At the other extreme, for ep waves $a_x a_y$ δ_{xy} are time independent so

$$\det (J) = a_x^2 a_y^2 - a_x^2 a_y^2 e^{J\delta_{xy}} e^{-J\delta_{xy}} = 0$$

$$[J] = \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \qquad PR - QQ^* = 0$$

Of particular interest is the decomposition of a partially polarised wave into the sum of an rp and and ep wave

$$J_{pp} = J_{rp} + J_{ep}$$

where

$$J_{pp} = \begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix} \qquad J_{rp} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad J_{ep} = \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix}$$

$$P,Q,R,I \geqslant 0 \qquad PR - QQ^* = 0$$

$$J_{xx} = I + P \qquad J_{xy} = Q$$

$$J_{yx} = Q^* \qquad J_{yy} = I + R$$

From the determinant equation

$$(J_{xx}-I)(J_{yy}-I) - J_{xy}J_{yx} = 0$$

Thus I is an eigenvalue of the coherency matrix given by

$$I = \tfrac{1}{2}(J_{xx} + J_{yy}) \pm \tfrac{1}{2}((J_{xx} + J_{yy})^2 - 4 \det (J))^{\frac{1}{2}}$$

Both roots of this equation are real and non-negative but since $P,R \geqslant 0$ the negative sign must be taken showing that the decomposition is unique. As before the total intensity of the wave is given by $Tr(J) = J_{xx} + J_{yy}$ and the total power density in the ep part of the decomposition

$$T_r(J_{ep}) = P + R = ((J_{xx} + J_{yy})^2 - 4 \det(J))^{\frac{1}{2}}$$

From this the degree of polarisation of the wave is defined as

$$D_p = \frac{\text{power in ep component}}{\text{total power}} = \left(1 - \frac{4 \det(J)}{(J_{xx}+J_{yy})^2}\right)^{\frac{1}{2}}$$

STOKES PARAMETERS

Of particular interest is the representation of the coherency properties by four real quantities instead of by the complex correlation matrix. One such choice of quantities was developed by Sir George Stokes in 1852 for use in vibrational theories of light propagation in the ether. They are formally defined[13] as the four components of the associated longitudinal vector of the polarisation spinor $\hat{P}$ as defined previously and as such allow the use of the Minkowski model of Lorentz space[14] for handling polarisation problems. They are related to the elements of the coherency matrix as follows:

$$g_0 = \tfrac{1}{2}(J_{xx} + J_{yy}) = \tfrac{1}{2}(a + d) = T_r(J) = \langle a_1^2\rangle + \langle a_2^2\rangle$$

$$g_1 = \tfrac{1}{2}(J_{xx} - J_{yy}) = \tfrac{1}{2}(a - d) = \langle a_1^2\rangle - \langle a_2^2\rangle$$

$$g_2 = \tfrac{1}{2}(J_{xy} + J_{yx}) = b = \langle a_1 a_2 \cos\delta_{12}\rangle$$

$$g_3 = i\tfrac{1}{2}(J_{xy} - J_{yx}) = c = \langle a_1 a_2 \sin\delta_{12}\rangle$$

$\underline{g} = (g_0\ g_1\ g_2\ g_3)$ is a Stokes vector

Physical interpretation may be placed on the elements of $\underline{g}$ by considering the case where the coherency matrix is represented in terms of (h,v) base. Under these circumstances g_0 represents the total power density of the wave; g_1 is a measure of how much like vertical or horizontal the wave polarisation is; g_2 a similar measure of $\pm 45^{\circ}$ nature and g_3 a measure of the ellipticity of the polarisation.

A derivation of the Stokes parameters from the coherency matrix may be had by considering an expansion of the latter in terms of the Pauli spin matrices[15] $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ given by

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These four matrices form a representation of a mathematical group with the following multiplication table:

	σ_0	σ_1	σ_2	σ_3
σ_0	σ_0	σ_1	σ_2	σ_3
σ_1	σ_1	σ_0	$-i\sigma_3$	$i\sigma_2$
σ_2	σ_2	$i\sigma_3$	σ_0	$-i\sigma_1$
σ_3	σ_3	$-i\sigma_2$	$i\sigma_1$	σ_0

It is well known(15) that any 2 x 2 matrix $[S]$ may be expanded in terms of this set so that

$$[S] = \sum_{\mu=0}^{3} S_\mu \, \sigma_\mu$$

where S_μ are the coefficients of expansion given by

$$S_\mu = \tfrac{1}{2} \, T_r\{S \, \sigma_\mu\}$$

For the particular case of the coherency matrix the expansion yields

$$S_0 = T_r(J\sigma_0) = a + d = g_0$$

$$S_1 = T_r(J\sigma_1) = b = g_2$$

$$S_2 = T_r(J\sigma_2) = c = g_3$$

$$S_3 = T_r(J\sigma_3) = a - d = g_1$$

Notice the change of order of Stokes parameters when compared with the previous definition, an unfortunate complication arising in many areas of polarisation algebra and one which causes untold confusion when carried through to scattering matrix theory. The above is known as the natural ordering of Stokes parameters and is frequently used in optics whereas the previous definition is termed traditional. Unfortunately in the radar literature(5) there is yet another permutation used and for want of a better name I shall call this the radar ordering. This arises from defining the Pauli matrices as

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

Notice that in this case S_μ must be multiplied by

$$[M]_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

in order to achieve real Stokes parameters.

In the radar ordering the Stokes vector takes the form

$$\underline{g} = (g_0, g_3, g_2, g_1)$$

To transform between the different systems the following matrices must be used

$$\begin{matrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{matrix}_{n} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{matrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{matrix}_{trad} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{matrix}_{radar}$$

Occasionally the so-called modified Stokes vectors are used which are related to $\underline{g} = (g_0, g_1, g_2, g_3)$ by

$$g_{m0} = g_0 + g_1 = \langle a_x^2 \rangle = a$$

$$g_{m1} = g_0 - g_1 = \langle a_y^2 \rangle = d$$

$$g_{m2} = g_2$$

$$g_{m3} = g_3$$

The orthogonality condition for Stokes vectors $\underline{g}_x$ and $\underline{g}_y$ is

$$\underline{g}_x \cdot \underline{g}_y = g_{0x}g_{0y} + g_{1x}g_{1y} + g_{2x}g_{2y} + g_{3x}g_{3y} = 0$$

Only the traditional ordering shall be considered in the remainder of this chapter since it leads most conveniently to the Poincaré sphere representation of polarisation.

Comparison of the properties of the Stokes vector with the coherency matrix gives rise to the following useful relationships.

The determinant condition for J becomes

$$g_0^2 \geq g_1^2 + g_2^2 + g_3^2$$

This is known as the condition for physical realisability of the Stokes vector and for an ep wave becomes

$$g_0^2 = g_1^2 + g_2^2 + g_3^2$$

For an rp wave

$$g_1 = g_2 = g_3 = 0 \quad \text{thus} \quad \underline{g} = (g_0, 0, 0, 0)$$

The decomposition theorem becomes

$$\underline{g} = \underline{g}_{ep} + \underline{g}_{rp}$$

$$\underline{g}_{ep} = ((g_1^2 + g_2^2 + g_3^2)^{\frac{1}{2}}, g_1, g_2, g_3)$$

$$\underline{g}_{rp} = (g_0 - (g_1^2 + g_2^2 + g_3^2)^{\frac{1}{2}}, 0, 0, 0)$$

Finally, the degree of polarisation is given by

$$D_p = \frac{(g_1^2 + g_2^2 + g_3^2)^{\frac{1}{2}}}{g_0}$$

The condition for physical realisability states that for an ep wave

$$g_0{}^2 = \sum_{i=1}^{3} g_i{}^2$$

The three Stokes parameters ($g_1\ g_2\ g_3$) may be thought of as the co-ordinates of a point in three dimensional space with distance from the origin given by g_0. Further, this radius is equal to the amplitude of the wave and as such the loci of polarisations of equal amplitude is a sphere. In optics this is called the Poinaré sphere, there being a one to one correspondence between points on the surface and the set of all possible polarisations. Note that this spherical geometry is only true for coherent scattering and for partially polarised waves the transformations occur on different spheres(16). It is a feature of the Stokes vector formalism that it allows the treatment of such problems and as such forms the basis for a general theory of depolarisation for which coherent scattering is a special case. Shown in Figure 3 are some examples of normalised Stokes vectors for some commonly used polarisations.

The transformation equations governing the movement of a point over the surface of a sphere are well known from trigonometry(17) and are best represented by three Euler angles, each representing a rotation about the x, y, z axis respectively. The geometry of this problem is shown in Figure 3 which shows a right handed cartesian co-ordinate system together with various co-ordinates of the point $\hat{P}$ in polarisation space. For transformation of $\hat{P}$ by an angle v about the x-axis the governing transformation matrix for $\underline{g}$ is

$$R_x(v) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{bmatrix}$$

Similarly for transformations by τ and ψ about the y and z axes respectively

$$R_y(\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\tau & 0 & -\sin\tau \\ 0 & 0 & 1 & 0 \\ 0 & \sin\tau & 0 & \cos\tau \end{bmatrix} \qquad R_z(\psi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A general transformation is then written

$$R = R_z(\psi)\, R_y(\tau)\, R_x(v)$$

Rather than consider the cartesian co-ordinates of $\hat{P}$ it is sometimes convenient to consider the angular co-ordinates (latitude and longitude). In order to see how these are related to the geometrical parameters of the polarisation ellipse consider again the original form of the coherency matrix.

$$[J] = \begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_x^* E_y \rangle & \langle E_y E_y^* \rangle \end{bmatrix}$$

In terms of the geometrical parameters

$$\underline{E} = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = a \begin{bmatrix} \cos\theta \cos\tau - i\sin\theta \sin\tau \\ \sin\theta \cos\tau + i\cos\theta \sin\tau \end{bmatrix}$$

Thus

$$\langle E_x E_x^* \rangle = \langle \frac{a^2}{2}(1 + \cos 2\theta \cos 2\tau) \rangle$$

$$\langle E_y E_y^* \rangle = \langle \frac{a^2}{2}(1 - \cos 2\theta \cos 2\tau) \rangle$$

$$\langle E_x E_y^* \rangle = \langle \frac{a^2}{2}(\sin 2\theta \cos 2\tau - i\sin 2\tau) \rangle$$

$$\langle E_y E_x^* \rangle = \langle \frac{a^2}{2}(\sin 2\theta \cos 2\tau + i\sin 2\tau) \rangle$$

Thus

$$s_0 = a^2$$

$$s_1 = a^2 \cos 2\theta \cos 2\tau$$

$$s_2 = a^2 \sin 2\theta \cos 2\tau$$

$$s_3 = a^2 \sin 2\tau$$

From standard spherical trigonometry these are seen to be the equations relating spherical polar to cartesian co-ordinates and hence the remarkable result that the latitude and longitude of a point $\hat{P}$ on the Poincaré sphere are 2τ and 2θ respectively where θ and τ are the inclination angle and ellipticity angle of the corresponding polarisation ellipse (Figure 4).

The relations between θ,τ and α,δ (the polarisation ratio parameters) are again standard results from spherical trigonometry[18]. They form the elements of a spherical triangle as shown in Figure 5. The α,δ co-ordinates figure prominently in the theory of null polarisations.

The Poincaré sphere has the following interesting properties:

a. The poles of the sphere represent left and right circular polarisations ($\tau = \pm 45^\circ$).

b. The upper and lower hemispheres map similar sets of elliptical polarisations with opposite sense. The upper hemisphere is chosen arbitrarily for left sense polarisations (positive τ).

c. The loci of polarisations of equal ellipticity are in planes parallel to the equator which itself represents the set of all linear polarisations.

d. Orthogonal polarisations lie diametrically opposite on the sphere.

It is worth noting the similarity between the Poincaré sphere and its plane projections[19] and a similar geometry used for impedance calculation in circuit theory. Indeed the Smith chart is a projection of the impedance sphere and may be similarly used to solve polarisation problems. This commonality between two seemingly different areas of work can be quite useful for understanding transformations on the sphere[19]. The similarity arises out of the fact that ρ is a complex ratio of like field components in a plane perpendicular to direction of propagation while the impedance Z is a similar ratio of electric and magnetic field components.

Although the Smith chart could be used for polarisation problems the most common plane projection used is the Polarisation Chart as shown in Figure 6. Note that two such charts are required to allow for mappings on both hemispheres, the chart being a projection of the sphere onto the equatorial plane so that linear polarisations lie around the circumference with circular in the centre. There has also been a scale change in order to give a linear reduction in eccentricity from unity in the centre to zero on the circumference.

RECEPTION OF POLARISED WAVES BY AN ANTENNA

In this section an expression will be derived for the power received by an antenna whose polarisation is fixed as $\hat{P}$, being used to receive a plane wave with polarisation $\hat{Q}$. An ideal antenna will be assumed for the sake of clarity and the more general problem of non-ideal antenna characteristics discussed in a later chapter.

The polarisation of an antenna is defined as the polarisation of the wave it radiates on transmit (remembering that this is defined in a right hand set of x,y,z co-ordinates where the positive z direction is pointing in the direction of propagation of the wave). Using the geometrical parameter representation we may then write the antenna polarisation as

$$\hat{P}_p = a_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \begin{bmatrix} \cos\tau_p \\ i\sin\tau_p \end{bmatrix}$$

'a_p' is the gain function of the antenna. The geometry of the problem to be solved is then as shown in Figure 7. The important point to note is that $\hat{Q}$ is defined in its own set of right handed co-ordinates which are different to those of $\hat{P}$. One system is obtained from the other by a 180° rotation about the y axis. In matrix notation the x,y co-ordinates may be related as

$$\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix}_{\hat{P}} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix}_{\hat{Q}} = \sigma_3\,\hat{Q}$$

Although it is desirable to define the polarisation of an incoming wave in terms of its own co-ordinate system it is also convenient to derive an expression for the received power using polarisations defined in the same co-ordinates since then the same point on the Poincaré sphere may be used to represent an antenna whether it transmits or receives.

In its own co-ordinate system the incoming wave is defined by

$$\hat{Q}_q = a_q\begin{bmatrix}\cos\theta_q & -\sin\theta_q\\ \sin\theta_q & \cos\theta_q\end{bmatrix}\begin{bmatrix}\cos\tau_q\\ J\sin\tau_q\end{bmatrix}$$

In order to express this in terms of the receiving antenna's co-ordinates

$$\hat{Q}_p = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\hat{Q}_q$$

$$= a_q\begin{bmatrix}\cos\theta_q\cos\tau_q - J\sin\theta_q\sin\tau_q\\ -\sin\theta_q\cos\tau_q - J\sin\tau_q\cos\theta_q\end{bmatrix}$$

It is easily verified that

$$\hat{Q}_q(a_q,\theta_q,\tau_q) = \hat{Q}_p(a_q,-\theta_q,-\tau_q)$$

Thus, if a wave of polarisation $\hat{Q}_q$ is incident upon an antenna, then for that antenna to be matched to the wave in the x,y plane it must not have the same polarisation as $\hat{Q}_q$ but the one given as $\hat{Q}_p$, ie opposite sense and negative inclination angle. This is called the symmetric polarisation to $\hat{Q}_q$ and is shown geometrically in Figure 8. This may seem a remarkable theorem and has certainly been at the root of much confusion in the past but is easily verified by considering transmission between two facing antennae, both of which are polarised at 45° linear. In this case they are orthogonally polarised to each other and so no power will be received at all. In fact it is only for vertical and horizontal that the symmetric polarisation is equal to its parent.

Also shown in Figure 8 are two other useful polarisations related to the same parent $\hat{Q}$. The conjugate polarisation is identical to $\hat{Q}_q$ except for a change of sense and is given by

$$\hat{Q}_c(a_c, \theta_c, -\tau_c)$$

The transverse polarisation has the same sense but negative the inclination angle

$$\hat{Q}_t(a_c, -\theta_c, \tau_c)$$

These polarisations are useful when considering the backscatter from targets with certain symmetries. Figure 9 shows how these polarisations relate to their parent on the Poincaré sphere. It is convenient that by adopting the co-ordinate change described we can dispense with such transformationson the sphere and write the equation for received voltage as

$$V = \hat{P} \cdot \hat{Q}_p = \hat{P} \cdot \hat{Q}_{Rq}$$

where $\hat{P}$ and $\hat{Q}$ are defined in their own co-ordinate systems and the dot product for complex vectors is defined as

$$\hat{a} \cdot \hat{b} = a_x e^{i\alpha_x} b_x e^{i\beta_x} + a_y e^{i\alpha_y} b_y e^{i\beta_y}$$

where

$$\hat{a} = \begin{bmatrix} a_x e^{i\alpha_x} \\ a_y e^{i\alpha_y} \end{bmatrix} \qquad \hat{b} = \begin{bmatrix} b_x e^{i\beta_x} \\ b_y e^{i\beta_y} \end{bmatrix}$$

Note that from this definition

$$\hat{a} \cdot \hat{a}^* = |a_x|^2 + |a_y|^2 = a^2$$

Before working out an explicit form for V it is worth noting a useful form of polarisation matrix algebra, based on the Pauli matrices discussed earlier, which allow a short hand approach to be used in solving what would otherwise be lengthy trigonometrical calculations. We know

$$\hat{p} = a\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\tau \\ i\sin\tau\end{bmatrix}$$

The θ matrix may be expanded in terms of the Pauli matrices to yield

$$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} = \cos\theta\ \sigma_0 - i\sin\theta\ \sigma_2 = e^{-\theta\sigma_2}$$

Similarly for the τ dependence

$$\begin{bmatrix}\cos\tau \\ i\sin\tau\end{bmatrix} = \begin{bmatrix}\cos\tau & i\sin\tau \\ i\sin\tau & \cos\tau\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} = e^{\tau\sigma_1}\begin{bmatrix}1 \\ 0\end{bmatrix}$$

And for the sake of completeness

$$\begin{bmatrix}e^{iv} & 0 \\ 0 & e^{-iv}\end{bmatrix} = \cos v\ \sigma_0 + i\sin v\ \sigma_3 = e^{v\sigma_3}$$

This extension of the exponential notation from ordinary complex number theory to Pauli matrices is a powerful technique and allows the general polarisation ellipse to be written as

$$\hat{p} = a\, e^{i\alpha}\, e^{-\theta\sigma_2}\, e^{\tau\sigma_1}\begin{bmatrix}1 \\ 0\end{bmatrix}$$

The following rules are useful for exploiting this algebra:

1. $e^{(\alpha+\beta)\sigma_2} = e^{\alpha\sigma_2} \cdot e^{\beta\sigma_2}$

Proof

$$e^{(\alpha+\beta)\sigma_2} = \cos(\alpha+\beta)\sigma_0 + i\sin(\alpha+\beta)\sigma_2$$

$$e^{\alpha\sigma_2} e^{\beta\sigma_2} = (\cos\alpha\sigma_0 + i\sin\alpha\sigma_2)(\cos\beta\sigma_0 + i\sin\beta\sigma_2)$$

$$= \cos\alpha \cos\beta + i\cos\alpha \sin\beta\sigma_2 + i\sin\alpha \cos\beta\sigma_2 - \sin\alpha \sin\beta$$

$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)\sigma_2$$

2. $\sigma_1 e^{\alpha\sigma_2} = e^{-\alpha\sigma_2} \sigma_1$

Proof

$\sigma_1(\cos\alpha\sigma_0 + i\sin\alpha\sigma_2) = (\cos\alpha\sigma_0 - i\sin\alpha\sigma_2)\sigma_1$ (see multiplication table)

3. $e^{\alpha\sigma_2} e^{\beta\sigma_1} \neq e^{\beta\sigma_1} e^{\alpha\sigma_2}$

but

$$e^{\alpha\sigma_2} e^{\beta\sigma_1} - e^{\beta\sigma_1} e^{\alpha\sigma_2} = 2 \sin\alpha \sin\beta\sigma_3$$

Using this notation the expression for received voltage

$$V = \hat{P} \cdot \ddot{Q}_s$$

$$= a_p e^{i\alpha_p} e^{-\theta_p\sigma_2} e^{\tau_p\sigma_1}\begin{bmatrix}1\\0\end{bmatrix} \cdot a_q e^{i\alpha_q} e^{\theta_q\sigma_2} e^{-\tau_q\sigma_1}\begin{bmatrix}1\\0\end{bmatrix}$$

$$= a_p a_q e^{i(\alpha_p+\alpha_q)} e^{-\tau_q\sigma_1} e^{-(\theta_p+\theta_q)\sigma_2} e^{\tau_p\sigma_1}\begin{bmatrix}1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\0\end{bmatrix}$$

$$= A e^{-\tau_q\sigma_1} (\cos(\theta_p+\theta_q)\sigma_0 - i \sin(\theta_p+\theta_q)\sigma_2) e^{\tau_p\sigma_1}\begin{bmatrix}1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\0\end{bmatrix}$$

And considering only the central portion

$$\cos(\theta_p+\theta_q)\, e^{(\tau_p-\tau_q)\sigma_1} + i\sin(\theta_p+\theta_q)\sigma_2\, e^{(\tau_p+\tau_q)\sigma_1}$$

put $\qquad \theta_p + \theta_q = \Sigma\theta \qquad \tau_p - \tau_q = \Delta\tau \qquad \tau_p + \tau_q = \Sigma\tau$

$$= \cos\Sigma\theta(\cos\Delta\tau\sigma_0 + i\sin\Delta\tau\sigma_1)\sigma_0 - i\sin\Sigma\theta\sigma_2(\cos\Sigma\tau\sigma_0 + i\sin\Sigma\tau\sigma_1)$$

$$= \cos\Sigma\theta \cos\Delta\tau\sigma_0 + i\cos\Sigma\theta \sin\Delta\tau\sigma_1 - i\sin\Sigma\theta \cos\Sigma\tau\sigma_2 + \sin\Sigma\theta \sin\Sigma\tau\, i\sigma_3$$

In matrix form this becomes

$$\begin{bmatrix} \cos\Sigma\theta \cos\Delta\tau - i\sin\Sigma\theta \sin\Sigma\tau & \sin\Sigma\theta \cos\Sigma\tau + i\cos\Sigma\theta \sin\Delta\tau \\ -\sin\Sigma\theta \cos\Sigma\tau + i\cos\Sigma\theta \sin\Delta\tau & \cos\Sigma\theta \cos\Delta\tau + i\sin\Sigma\theta \sin\Sigma\tau \end{bmatrix}$$

and when multiplied out the expression for V becomes

$$V = A(\cos(\theta_p+\theta_q) \cos(\tau_p-\tau_q) - i\sin(\theta_p+\theta_q) \sin(\tau_p+\tau_q))$$

The received power is given by VV^*

$$P = |V|^2 = VV^* = |A|^2(\cos^2(\theta_p+\theta_q)\cos^2(\tau_p-\tau_q) + \sin^2(\theta_p+\theta_q)\sin^2(\tau_p+\tau_q)$$

and by using the relationships

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos2\theta)$$

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos2\theta)$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-b) = \cos A \cos B - \sin A \sin B$$

this becomes

$$P = \frac{a_p^2 a_q^2}{2}[1 + \sin2\tau_p \sin2\tau_q + \cos2\tau_p \cos2\theta_p \cos2\tau_q \cos2\theta_q - \cos2\tau_p \sin2\theta_p \cos2\tau_q \sin2\theta_q]$$

This is a fundamental equation in polarisation analyses and as such warrants further comment. Note that for maximum power received the conditions

$$\theta_p = -\theta_q \quad \text{and} \quad \tau_p = \tau_q \quad \text{must be satisfied.}$$

In this case

$$P = a_p^2 a_q^2$$

Notice that the condition on the ellipticity is not as might be expected since it states that maximum power is received when the incident wave has the same sense as the antenna, when both are quoted in their own co-ordinate system. This is due to the fact that the sense of polarisation changes not only with the rotation of co-ordinates provided by σ_3 but also depending on the direction of observation, ie either parallel or antiparallel to the direction of propagation.

There are four independent power measurements required by an antenna in order to determine $\hat{Q}$ as evidenced by the equation for P. Usually the four used are linear vertical and horizontal, +45° linear and a circular polarisation[21] and this allows the determination of $(\theta_q\ \tau_q)$. Alternatively a dual channel receiver could be used measuring simultaneously the vertical and horizontal components of $\hat{Q}$ and as long as some measure of the time phase angle between V and H is provided this allows for the instantaneous measurement of polarisation. This latter technique will be used later in order to determine the scattering from targets.

In terms of the Stokes parameters the equation for P takes on the form of a dot product, namely

$$P = g_0h_0 + g_1h_1 - g_2h_2 + g_3h_3$$

where

$$\underline{g} = (g_0\ g_1\ g_2\ g_3) \quad - \quad \text{Stokes vector for } \hat{Q}\text{, the incident wave}$$

$$\underline{h} = (h_0\ h_1\ h_2\ h_3) \quad - \quad \text{Stokes vector for } \hat{P}\text{, the antenna polarisation}$$

Notice that this may be written in matrix form as

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} g_o \\ g_1 \\ g_2 \\ g_3 \end{matrix} \cdot \begin{matrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{matrix} = [M]\, \underline{g} \cdot \underline{h}$$

In general the received power P_r will not be a maximum and this leads to the definition of the polarisation efficiency

$$\Gamma = \frac{P_r}{a_p^2 a_q^2}$$

$\Gamma = 1$ for polarisation matched antenna

$\Gamma = 0$ for orthogonally polarised antenna

On the Poincaré sphere $\hat{P}$ may be mapped as the antenna polarisation and then $\hat{P}_\perp$ lies diametrically opposite. If $\hat{Q}$ is the polarisation of the incident wave the polarisation efficiency may be related quite simply to the angular separation between $\hat{P}$ and $\hat{Q}$ as follows. Figure 10 shows the two polarisations mapped on the sphere with 2ψ the angular separation. Figure 11 shows the so-called power density semicircle from which it is evident that the normalised power in $\hat{Q}$ is

$$\cos^2\psi + \sin^2\psi = 1$$

If $\hat{Q}$ is then decomposed into $(\hat{P}\ \hat{P}_\perp)$, the fractional power contained in the $\hat{P}$ and $\hat{P}_\perp$ components is $\cos^2\psi$ and $\sin^2\psi$ respectively. From the definition of Γ this means

$$\Gamma = \cos^2\psi$$

This relationship is of great interest in polarisation filter design since it clearly demonstrates that the loci of constant response are circles centred on the antenna polarisation. One of the reasons the polarisation chart is used as a plane projection is that circles on the sphere translate to circles on the chart and so allow the design of filters with the desired response in polarisation space(7).

TARGET SCATTERING MATRIX

In general terms a radar target may be described as a polarisation transformer in that it operates on the incident wave such that the polarisation of the reflected wave bears a complicated but deterministic relationship to the incident polarisation. Further, the nature of this transformation will depend on the geometry, surface structure and material composition of the target and in the remaining part of these lectures it will be shown how by making appropriate field measurements and processing the data in the correct way this information may be made available for enhanced target detection and target identification.

For a complex radar target like an aircraft the scattering will not only be a function of incident polarisation but of target aspect and radar frequency. It will be assumed initially that these are fixed and that the scattered field parameters are linearly related to the incident field. The polarisation transforming properties may then be expressed as a transformation matrix, the polarisation scattering matrix. The dimensions and form of this matrix depend on the representation of polarisation used: for coherent scattering the incident and reflected waves may be expressed in terms of polarisation base (x,y) and the matrix is then 2 x 2 complex (Figure 12). Note that in general eight measurements will be required; four amplitudes and four phase angles, for the determination of this matrix.

If the Stokes parameters had been used then the scattering matrix would be 4 x 4 and real (Figure 12). This then requires sixteen amplitude measurements to be made but will contain extra information above that provided by the coherent matrix since it is based on the more general Stokes vector formalism. This matrix is known as the Stokes reflection or Mueller matrix and is of fundamental importance in describing the scattering of partially polarised waves. In 1970, Huynen[5] developed a decomposition theorem analagous to the decomposition of the coherency matrix enabling a set of average coherent scattering matrix parameters to be obtained from the Stokes matrix. In this chapter we shall consider only the coherent matrix and its properties.

For [S] in (h,v) we may write

$$\begin{bmatrix} E_h \\ \\ E_v \end{bmatrix}_{scattered} = A \begin{bmatrix} |S_{HH}| \; e^{J\phi_{HH}} & |S_{HV}| \; e^{J\phi_{HV}} \\ \\ |S_{VH}| \; e^{J\phi_{VH}} & |S_{VV}| \; e^{J\phi_{VV}} \end{bmatrix} \begin{bmatrix} E_h \\ \\ E_v \end{bmatrix}_{transmit}$$

$$\mathbf{E}_s = [S] \; \mathbf{E}_t$$

It is assumed that $[S]$ in this form has been premultiplied by σ_3 so that the h,v axes are the same for transmit and receive as required by the Poincaré sphere representation. In the notation employed S_{xy} means transmitting polarisation $\hat{x}$ and receiving polarisation $\hat{y}$ in both amplitude and phase.

The voltage received at a pair of antenna terminals for fixed transmit polarisation $\hat{P}_T$, receiver polarisation $\hat{P}_R$ and target scattering matrix $[S]$ may be written

$$V = \hat{P}_R \cdot [S] \; \hat{P}_T$$

For monostatic radar systems (ie ones using the same antenna for transmit and receive) various simplifications may be made to $[S]$.

The absolute phase of the target is a function of its range and velocity and as such is not a target related parameter (unless doppler type analysis is required). Therefore there is no loss of generality if one of the phase angles in $[S]$ is set to zero and the others measured relative to the corresponding phase centre on the target. This then reduces the number of required measurables to seven.

For monostatic systems the reciprocity theorem for antennas demands

$$V = \hat{P}_A \cdot [S] \; \hat{P}_B = \hat{P}_B \cdot [S] \; \hat{P}_A$$

but

$$\hat{P}_A \cdot [S] \; \hat{P}_B = [S]^T \; \hat{P}_A \cdot \hat{P}_B = \hat{P}_B \cdot [S]^T \; \hat{P}_A$$

which implies

$$[S]^T = [S]$$

This means that for monostatic systems the scattering matrix is symmetric

ie $\qquad S_{HV} = S_{VH}$.

If the phase of these diagonal terms is taken as reference then the so-called relative scattering matrix results

$$\begin{bmatrix} E_h \\ E_v \end{bmatrix}_{receive} = A \begin{bmatrix} |S_{HH}|\, e^{J\phi_{HH}-\phi_{HV}} & |S_{HV}| \\ |S_{VH}| & |S_{VV}|\, e^{J\phi_{VV}-\phi_{HV}} \end{bmatrix} \begin{bmatrix} E_h \\ E_v \end{bmatrix}_{transmit}$$

This matrix is determined by measurement of five parameters, namely three amplitudes and two relative phase angles.

Occasionally another 2 x 2 target matrix is used, called Grave's Power matrix it is related to [S] as follows:

$$E_s = [S]\, E_t$$

The scattered power is given by

$$E_s^* \cdot E_s = E_t^{*T} [S]^{*T} [S]\, E_t = E_t^{*T} P\, E_t$$

$$P = S^{*T} S = \begin{bmatrix} a & c \\ c^* & b \end{bmatrix}$$

where S^{T*} is the transpose conjugate of [S] and P is Hermitian. For example, in (h,v)

$$a = |S_{HH}|^2 + |S_{HV}|^2$$

$$b = |S_{VV}|^2 + |S_{VH}|^2$$

$$c = S_{HH}^* S_{HV} + S_{VH}^* S_{VV}$$

Thus 'a' is the total available backscattered power in the horizontal component due to both the S_{HH} and S_{HV} elements of [S]. Measurement of this matrix may be had by transmitting four different polarisations and receiving the total power backscattered in each case. This matrix contains a subset of the target information contained in [S] and so will not be considered further in these lectures.

Figure 13 shows some examples of the relative phase scattering matrix for simple radar targets. Note that all these matrices are quoted in (h,v).

The identity matrix is indicative of reflection from a flat plate at normal incidence or from a trihedral retroreflector (both are same in a polarimetric sense) while the double bounce dihedral reflector has a 180° phase difference between HH and VV. Also shown are some other important scattering types such as the linear target and helix both of which are important when considering the classification of targets as will be covered later in this course.

The most powerful aspect of scattering matrix measurement is that it allows the prediction of target scattering for any transmit polarisation and as such provides full polarimetric information about the target.

The transformation equations governing the prediction of target backscatter in base (x,y) when it is known in (x',y') may be developed using the change of base matrix [T] derived earlier. Care must be taken however to remember that [S] is an operator relating an incident to a backscattered wave and as such, co-ordinate changes must be taken into account. The following relationships hold:

1. [T] is a unitary matrix operator with unit determinant

$$[T]^{-1} = [T]^{*T} \qquad \det(T) = 1$$

2. For the incident system

$$\underline{E}_t(x,y) = [T]\ \underline{E}_t(x',y')$$

3. For the scattered system

$$\underline{E}_s(x,y) = [T]^*\ \underline{E}_s(x',y')$$

The conjugate operator is employed because there is a change of sense of polarisation when the scattered co-ordinate system is observed from the transmitter co-ordinates.

From the definition of the scattering matrix

$$\underline{E}_s(x,y) = [S(x,y)]\ \underline{E}_t(x,y)$$

$$= [S(x,y)][T]\ \underline{E}_t(x',y')$$

From 3

$$\underline{E}_s(x',y') = [T]^T\ \underline{E}_s(x',y')$$

$$\therefore \quad \underline{E}_s(x',y') = [T]^T[S(x,y)][T]\ \underline{E}_t(x,y)$$

Thus in general the transformation of [S] is given by the congruent transformation

$$[S(x',y')] = [T]^T[S(x,y)][T]$$

$$[T] = \frac{1}{\sqrt{1+\rho\rho^*}}\begin{bmatrix} 1 & -\rho^* \\ \rho & 1 \end{bmatrix}$$

By putting

$$[S] = \begin{bmatrix} S_{XX} & S_{XY} \\ S_{YX} & S_{YY} \end{bmatrix}$$

and expanding the matrix product, the transformation equations become

$$S_{X'X'} = (1+\rho\rho^*)^{-1} [\rho^2 S_{YY} + \rho(S_{XY} + S_{YX}) + S_{XX}]$$

$$S_{X'Y'} = (1+\rho\rho^*)^{-1} [\rho\ S_{YY} - \rho^* S_{XX} + S_{XY} - \rho\rho^* S_{YX}]$$

$$S_{Y'X'} = (1+\rho\rho^*)^{-1} [\rho\ S_{YY} - \rho^* S_{XX} + S_{YX} - \rho\rho^* S_{XY}]$$

$$S_{Y'Y'} = (1+\rho\rho^*)^{-1} [\rho^{*2} S_{XX} + S_{YY} - \rho^*(S_{XY} + S_{YX})]$$

For monostatic scattering these simplify to

$$S_{X'X'} = (1+\rho\rho^*)^{-1} [\rho^2 S_{YY} + 2S_{XY}\rho + S_{XX}]$$

$$S_{X'Y'} = (1+\rho\rho^*)^{-1} [\rho S_{YY} - \rho^* S_{XX} + S_{XY}(1-\rho\rho^*)]$$

$$S_{Y'X'} = (1+\rho\rho^*)^{-1} [\rho S_{YY} - \rho^* S_{XX} + S_{XY}(1-\rho\rho^*)]$$

$$S_{Y'Y'} = (1+\rho\rho^*)^{-1} [\rho^{*2} S_{XX} + S_{YY} - \rho^*\ 2S_{XY}]$$

Note that the symmetry of the monostatic matrix is preserved under this transformation. The following quantities are also invariant under change of base:

a. $\det([S(x',y')]) = \det([T]^T) \det S(x,y) \det T = \det S(x,y)$

b. $T_r(P)$ = total power returned to a pair of orthogonally polarised antennas = Span([S]) = invariant.

These equations are quadratics in ρ, where ρ relates $(x'y')$ to (x,y). For example, if [S(h,v)] is known then [S(left circular, right circular)] may be predicted by setting $\rho = J$. Then

$$[S(\ell,r)] = \begin{bmatrix} \frac{S_{HH}-S_{VV}}{2} + J\ S_{HV} & J\ \frac{S_{HH}+S_{VV}}{2} \\ J\ \frac{S_{HH}+S_{VV}}{2} & \frac{S_{VV}-S_{HH}}{2} + J\ S_{HV} \end{bmatrix}$$

Substituting the values for the trihedral gives

$$S(\ell,r) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and for the dihedral

$$S(\ell,r) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This confirms the fact that for each reflection off a metallic surface, the sense of polarisation is changed.

The congruent nature of the change of base transformation has several important implications for interpretation of polarimetric data and in particular the following theorems are important:

1. Theorem 1: Any square matrix [S] can be transformed to diagonal form by unitary matrices M and Q such that

$$MSQ = D = \alpha_i \, \delta_{iJ} \qquad \delta_{iJ} \text{ - Dirac delta function}$$

Proof

$$[MSQ]^{*T} = Q^{*T}S^{*T}M^{*T} = \alpha_i^{*}\delta_{iJ}$$

Now premultiply

$$Q^{*T}S^{*T}M^{*T}MSQ = Q^{*T}S^{*T}SQ = \alpha_i\alpha_i^{*}\delta_{iJ}$$

Since Q is unitary

$$Q^{-1} = Q^{*T}$$

Thus Q is the unitary matrix which diagonalises the Hermitian product $S^{*T}S$ by a similarity transformation.

By post-multiplying we have

$$MSQ\,Q^{*T}S^{*T}M^{*T} = MSS^{*T}M^{*T} = \alpha_i\alpha_i^{*}\delta_{iJ}$$

Thus M is the unitary matrix which diagonalises the Hermitian product SS^{*T} by a similarity transformation.

If S is assumed to be symmetric then $M = Q^T$

Proof

$$MSQ = D$$

$$Q^TS^TM^T = D^T = Q^TSM^T = D$$

$$\therefore \; M = Q^T$$

Hence we arrive at the very important result that under the change of base transformation the coherent scattering matrix can always be diagonalised

$$\text{ie} \quad [T]^T[S][T] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

This is a result of fundamental importance since it implies the existence of polarisations which remain unchanged when incident upon the target. These so-called eigenpolarisations form the basis for the processing techniques to be described later.

2. Theorem 2: The power matrix P is diagonalised by a similarity transformation using the same unitary matrix Q which diagonalised the scattering matrix by a congruent transformation and the eigenvalues of P are real and equal to the squared magnitude of the eigenvalues of [S]

Proof $\quad P = S^{T*}S$ and so by 1. the theorem is proved.

MEASUREMENT OF THE SCATTERING MATRIX

There have been very many different techniques proposed for measuring the target scattering matrix and a good review of these is provided by Huynen[21]. In this section only the basic requirements will be outlined and some of the limitations and problems pointed out.

The scattering matrix with absolute phase may be measured using a coherent radar with a dual channel receiver capable of simultaneously receiving two orthogonal polarisations and providing I and Q outputs for each channel (Figure 15). The transmitter must then transmit two orthogonal polarisations such that the four complex elements of [S] may be measured. Ideally, the whole matrix should be measured at the same time but this requires some form of coding for the two orthogonal transmitted polarisations so that the matrix elements can be separated on receive. The easiest way of doing this is to time multiplex the transmit waveform such that in a pulsed radar the transmit polarisation is changed on a pulse to pulse basis. Thus it takes 2 PRI to measure the matrix but isolation of the matrix elements is guaranteed. Note that if a priori knowledge is available about the targets to be measured then it may be possible to reduce these requirements but in general this will not be so.

The two main components needed for the SM radar above those of a conventional fixed polarisation device are a polariser in the transmit channel and an ortho-mode coupler (OMT) in the receiver. The latter device is a standard microwave component and allows the separation of an incident elliptically polarised wave into two orthogonal components. The polariser may take many forms such as a mechanical switch between vertically and horizontally polarised feed channels or a single channel Faraday rotation device with a ferrite phase shifter for generating elliptical polarisations. Whatever the configuration, the transmitter must be able to switch alternate polarisations at the PRF rate. This brings into play an important problem in the measurement of scattering matrix. If the target moves during the 2 PRI time-period necessary to measure the matrix then the measurement will be inaccurate and if movement is too severe and the time period too long then phase determination may be impossible. Thus coherent measurement of the matrix is reserved usually for carefully staged range measurements or for high PRF radars where the target motion is known to be within a certain bandwidth less than the PRF.

The problem may be overcome by measuring the relative scattering matrix, since then, absolute movement of the target over the period of matrix determination is less critical. A schematic diagram of a pulsed radar suitable for measurement of the relative SM is shown in Figure 14 together with a typical measurement switching programme. The essential difference between this method and the previous one is in the measurement of phase angle. This latter technique does not require a coherent radar since the PSD may take one of the received channels as its phase reference and so provide a measure of the time phase difference between the two orthogonal polarisation components. The PSD must be able to unambiguously determine angles in the range

$$-180^{\circ} \leq \phi_{PSD} < 180^{\circ}$$

One interesting possibility is to mix bases, ie transmit on base (x,y) but receive base (x',y'). If the relationship between these bases is known as

$$\underset{\sim}{E}(x,y) = [T]\ \underset{\sim}{E}(x',y')$$

then the measured matrix [Q] may be converted into the same base by

$$E(x',y') = [Q]\ E(x,y)$$

$$= [Q][T]\ E(x',y')$$

This technique may be of use when there are limited dynamic range problems in the radar receiver.

To date, measurement has been considered using ideal antennae and a noise free system. In order to consider a more realistic system it is necessary to be able to quantify errors introduced by using real antennas which have finite losses and cross polar isolation. In order to do this one may consider a transmission matrix for the antenna. In other words if polarisation vector $\hat{P}$ is requested for transmission, what will actually be transmitted is

$$\hat{P}' = [E]\ \hat{P}$$

For an ideal system [E] will be the identity matrix but for real systems will represent a distortion of ellipticity $\Delta\tau$ and a rotation of the plane of polarisation

$\Delta\theta$. If losses within the antenna are assumed small then [E] will be of the form

$$[E] = \begin{bmatrix} \cos\Delta\theta & -\sin\Delta\theta \\ \sin\Delta\theta & \cos\Delta\theta \end{bmatrix} \begin{bmatrix} \cos\Delta\tau & J\sin\Delta\tau \\ J\sin\Delta\tau & \cos\Delta\tau \end{bmatrix} = \begin{bmatrix} e^{\varepsilon_{11}} & \varepsilon_{12} \\ \varepsilon_{21} & e^{\varepsilon_{22}} \end{bmatrix}$$

where ε_{11}, ε_{22}, ε_{12}, ε_{21} are complex and $Re(\varepsilon_{11}) \leq 0$, $Re(\varepsilon_{22}) \leq 0$, the equality holding for zero losses. ε_{iJ} are functions of position within the antenna beam and as one deviates further from boresight these errors tend to increase.

On receive the matrix $[E]^*$ operates on the desired polarisation $\hat{Q}$, so the net effect on measurement of [S] is the congruent transformation

$$[S]^1 = [E]^T[S][E] \qquad\qquad [E] \text{ - unitary for lossless antenna}$$

In reality losses may arise from heat loss, reflections from support structures and radomes. In this case [E] is non-unitary and the determinant and Trace of [S] are no longer invariant under the above transformation. Using the above form for [E] and putting

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

yields

$$[S]' = e^{\varepsilon_1} \begin{bmatrix} e^{\varepsilon_2} S_{11} + \varepsilon_{21} S_3 & S_{12} + \varepsilon_{12}S_{11} + \varepsilon_{21}S_{22} \\ S_{21} + \varepsilon_{12}S_{11} + \varepsilon_{21}S_{22} & e^{-\varepsilon_2} S_{22} + \varepsilon_{12} S_3 \end{bmatrix}$$

where

$$\varepsilon_1 = \varepsilon_{11} + \varepsilon_{22}$$

$$S_3 = S_{12} + S_{21}$$

$$\varepsilon_2 = \varepsilon_{11} - \varepsilon_{22}$$

Notice that if S_{12}, S_{21} are small relative to the copolar terms then because ε_{12} ε_{21} are small, the copolar elements are less susceptible to error. The cross-polar terms however are more inaccurate.

In order to try and calibrate these errors in a system then at least two calibration targets are required with known scattering matrices. For example

$$[S] = A\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow S' = Ae^{\varepsilon_1}\begin{bmatrix} 2\,\varepsilon_{21} & 1 \\ 1 & 2\,\varepsilon_{12} \end{bmatrix}$$

hence can get ε_{12}, ε_{21}, which relate to the cross-polar isolation of the antenna. If

$$[S] = A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow S' = Ae^{\varepsilon_1}\begin{bmatrix} e^{\varepsilon_2} & \varepsilon_{12} + \varepsilon_{21} \\ \varepsilon_{12} + \varepsilon_{21} & e^{-\varepsilon_2} \end{bmatrix}$$

and if the absolute RCS of calibrator is known the ε_1, ε_2 may be determined. Two targets which correspond to these calibration matrices are a trihedral and dihedral with its seam at 45° to the radar vertical. If these calibrations are made and the scattering matrix measured within the target's scattering centre decorrelation time reliable SM data may be collected and made available for processing. Note that if the Stokes parameters are used then they give a direct measure of the polarisation purity of the received wave and its susceptibility to noise.

TARGET PARAMETERS

By measurement of the five parameters of the relative phase SM in some base (x,y), the scattering for any incident polarisation, given by the loci of points on the surface of the Poincaré sphere, may be predicted by the transformation equations derived earlier. The question arises as to how best process this data in a manner which will yield information relating to target geometry and symmetry independent of which base is chosen for measurement. It will be shown in this section how such a set of five target parameters may be derived and how they may be used to improve radar performance.

The target descriptors are based on knowledge of the so-called target characteristic or null polarisations. These are transmit/receive polarisation pairs which result in zero target backscatter. There are in general four such polarisations and every target, however complex, has such a set. The proof of the existence of these polarisations is based on theorem 1 from the previous section, namely that there always exist two polarisation vectors which diagonalise the symmetric scattering matrix.

The two polarisations corresponding to the diagonalisation of [S] are termed the eigenpolarisations or cross-polar nulls (XPOL) in that when they are transmitted they remain unaltered on reflection, so that by receiving in the orthogonal channel a null response would be obtained. Care must be taken when calculating these polarisations since as usual the co-ordinate frames must be related so that the Poincaré sphere representation may be used. When allowance is made for the conjugate nature of the backscatter co-ordinates when compared to the transmitter then the eigenpolarisations are solutions of

$$[S]\, \underline{E}_m = \lambda \underline{E}_m^* \qquad \lambda \text{ - complex eigenvalues}$$

It is now apparent why these are called eigenpolarisations since this equation is similar to the classical eigenvalue problem

$$[S]\, \underline{E} = \lambda \underline{E}$$

The conjugate sign in the polarisation problem forces the corresponding eigenvalue to be phase determined whereas in the more commonly met form of eigenvalue problem, if λ is a solution then so is $\lambda e^{i\delta}$ $0 \leq \delta < 360^{\circ}$.

From theorem 2 the eigenvectors of [S] are the same as those of S^*S and the latter has real eignvalues given by the square moduli of the eigenvalues of [S]. When the eigenvalues of [S] are distinct then it may be shown that the two corresponding eigenvectors are always orthogonal.

<u>Proof</u> let λ_1, λ_2 be the two eigenvalues and $\lambda_1 > \lambda_2$

then $$[S]\, \underline{E}_1 = \lambda_1 \underline{E}_2^*$$

$$[S]\, \underline{E}_2 = \lambda_2 \underline{E}_2^*$$

Since [S] is symmetric

$$[S]\underline{E}_1 \cdot \underline{E}_2 = \underline{E}_1 \cdot [S]\underline{E}_2$$

$$|\lambda_1| \; |\underline{E}_1^* \cdot \underline{E}_2| = |\lambda_2| \; |\underline{E}_1 \cdot \underline{E}_2^*|$$

hence if $\lambda_1 \neq \lambda_2$

$$\underline{E}_1 \cdot \underline{E}_2^* = 0$$

This is the condition for othogonality of polarisation vectors.

On the Poincaré sphere $\underline{E}_1, \underline{E}_2$ lie diametrically opposite and so knowledge of one immediately determines the other. Thus two of our desired target parameters are the latitude and longitude or ellipticity and inclination angle of one of the eigenpolarisations. The one chosen is the eigenvector with the largest corresponding eigenvalue since the maximum RCS of the target is given by the square modulus of the maximum eigenvalue.

Proof

$$P_{TOT} = [D]\underline{E} \cdot [D]^*\underline{E}^* \qquad [D] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$= (a[D]\underline{E}_1 + b[D]\underline{E}_2)(a^*[D]^* \underline{E}_1^* + b^*[D]^* \underline{E}_2^*)$$

$$= |a|^2|\lambda_1|^2 + |b|^2|\lambda_2|^2 \qquad \text{since } \underline{E}_i \, \underline{E}_j^* = \delta_{ij}$$

$$D\underline{E}_1 = \lambda_1 \underline{E}_1^*$$

$$= |\lambda_1|^2 - |b|^2(|\lambda_1|^2 - |\lambda_2|^2)$$

Hence if $|\lambda_1| > |\lambda_2|$ then maximum power is obtained if $|b| = 0$ ie the maximum polarisation $\hat{E}_{max} = \hat{E}_1$ and $P_{max} = |\lambda_1|^2$. In the above

$$\underline{E} = a\underline{E}_1 + b\underline{E}_2 \quad \text{and} \quad |a|^2 + |b|^2 = 1$$

This maximum polarisation is the third target parameter and is obviously of great importance to radar systems since it is the maximum RCS that may be obtained for a target at a fixed aspect and frequency.

Another method of calculating the eigenpolarisations is to consider the general transformation equations, quadratic in ρ. The eigenpolarisations are seen to be solutions to

$$S_{XY} = 0 = \rho S_{YY} - \rho^* S_{XX} + S_{XY}(1 - \rho\rho^*)$$

This equation may be solved by noting that S^*S has the same eigenvectors as [S] and is diagonalised by a similarity transformation

$$T^{-1} S^* S T$$

If this matrix product is expanded then for the off-diagonal terms to be zero the equation

$$\rho^2 b + \rho(a-c) - b^* = 0$$

where

$$a = S_{XX}S_{XX}^* + S_{XY}S_{XY}^*$$

$$b = S_{XY}S_{XX}^* + S_{YY}S_{XY}^*$$

$$c = S_{YY}S_{YY}^* + S_{XY}S_{XY}^*$$

so

$$\rho_{1,2} = \frac{R_1 \pm \sqrt{R_1^2 + 4R_2R_3}}{2R_2}$$

$$R_1 = |S_{YY}|^2 - |S_{XX}|^2$$

$$R_2 = S_{YY}S_{XY}^* + S_{XY}S_{XX}^*$$

$$R_3 = R_2^*$$

These equations give two values of ρ corresponding to the two eigenpolarisations. By writing $\rho = \tan\alpha\, e^{J\delta}$ then the latitude and longitude of the eigenpolarisations may be found from

$$\theta_1 = \tfrac{1}{2} \arctan(\tan 2\alpha \cos\delta)$$

$$\tau_1 = \tfrac{1}{2} \arcsin(\sin 2\alpha \sin\delta)$$

As a simple example of the eigenpolarisations of a target consider the backscatter from a wire grid target as shown in Figure 16. If the grid is aligned at 0° to the incident vertical then the polarisation of the backscattered wave will be vertical, independent of the incident polarisation. Thus vertical is one eigenpolarisation while horizontal is the other. Further, vertical is obviously the maximum polarisation since for horizontally polarised waves there is zero backscatter. If the grid is now rotated about the radar line of sight then these eigenpolarisations remain linear but rotate in the equatorial plane of the Poincaré sphere.

The remaining two characteristic polarisations are the copolar nulls (COPOL) or incident polarisations which are transformed into their orthogonal state on reflection. These may be determined from the transformation equations by

$$S_{XX}' = 0 = \rho^2 S_{YY} + 2S_{XY}\rho + S_{XX}$$

ie $$\rho_{3,4} = \frac{-S_{XY} \pm \sqrt{S_{XY}^2 - S_{XX}S_{YY}}}{2S_{YY}}$$

It is these polarisations which are exploited in rain clutter suppression radars since spherical raindrops have left and right circular polarisation as their copolar nulls. Notice that, although in this case the COPOL nulls are orthogonal, in general they will not be so.

Of particular importance is the relation of the COPOL nulls to the eigenpolarisations and in order to develop this relationship consider transformation of the scattering matrix when it is expressed in (X_1, X_2) where $\hat{X}_1$ and $\hat{X}_2$ are the eigenpolarisations

$$[S]' = [T]^T [D] [T]$$

$$= \frac{1}{1+\rho\rho^*} \begin{bmatrix} 1 & \rho \\ -\rho^* & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\rho^* \\ \rho & 1 \end{bmatrix} \qquad \rho = \frac{\hat{X}_2}{\hat{X}_1}$$

$$= \frac{1}{1+\rho\rho^*} \begin{bmatrix} \lambda_1 + \rho^2\lambda_2 & \rho\lambda_2 - \rho^*\lambda_1 \\ \rho\lambda_2 - \rho^*\lambda_1 & \lambda_2 + \rho^{*2}\lambda_1 \end{bmatrix}$$

For the transformed matrix to be a COPOL null then ρ must satisfy

$$\lambda_1 + \rho^2\lambda_2 = 0$$

$$\rho_{3,4}^2 = -\frac{\lambda_1}{\lambda_2} = \tan^2\alpha \, e^{J2\delta}$$

From this we may write $[D]$ in the form

$$D = \begin{bmatrix} m \, e^{J(\phi+v)} & 0 \\ 0 & m \tan^2\gamma \, e^{J(\phi-v)} \end{bmatrix} \quad \text{where } m = |\lambda_1|$$

$$\gamma = 90^\circ - \alpha \qquad 0 < \gamma < 45^\circ$$

$$v = \delta - 90^\circ \qquad -90^\circ < v < 90^\circ$$

The two solutions satisfying these conditions are

$$\rho_3 = +J\sqrt{\frac{\lambda_1}{\lambda_2}} = \tan\alpha \, e^{J\delta}$$

$$\rho_4 = -J\sqrt{\frac{\lambda_1}{\lambda_2}} = -\tan\alpha \, e^{J\delta}$$

In terms of the Deschamps co-ordinates on the Poincaré sphere these points are given by $(2\gamma, v)$ and $(-2\gamma, v)$ taken from the minimum eigenpolarisation $\hat{X}_2$. In order to express the transformations from the maximum polarisation $\hat{X}_1$, then

$2\gamma \rightarrow 180^{\circ} - 2\gamma$ and $v \rightarrow -v$. Hence the points become $(2\alpha,v)$ and $(-2\alpha,v)$, where positive v is taken in an anticlockwise direction from the equatorial plane and γ, v are angles from [D].

These results are very important since they show an interesting geometrical relationship between the two sets of null polarisations. As has already been noted the eigenpolarisations lie diametrically opposite and now we have the additional information that the COPOL nulls have co-ordinates $(\pm 2\alpha, -v)$ from the maximum eigenpolarisation. Thus all four points lie on one great circle which defines a plane in polarisation space (Figure 17) and the diameter joining the eigenpolarisations bisects the planar angle between the COPOL nulls. The resulting structure is known as the polarisation fork and is a powerful graphical aid to radar target classification studies. The fork prongs are drawn from the origin to the COPOL nulls and the length of the 'handle' is the radius of the Poincaré sphere or the span ([S]), which is a transformation invariant equal to the power in the polarised component of the wave. Note that some authors take the target maximum RCS as radius of the sphere but in order that the results be consistent with the Stokes formalism it will be taken in these lectures as

$$r = g_1^2 + g_2^2 + g_3^2 = S_p([S])$$

This now enables us to classify different radar targets according to a five element target vector given by

$$V = (2\theta_1, 2\tau_1, v, \gamma, m)$$

The first three angles define the plane of the fork in three dimensions, the fourth angle is γ being the fork separation and relating the difference in magnitude of target eigenvalues. The final parameter m is the target maximum RCS. In the next section these ideas will be developed further and examples given but first consider again the diagonalised form of [S]

$$[S] = T^* D T^{T*} \qquad [D] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} m\ e^{i\phi+v} & 0 \\ 0 & m \tan^2\gamma\ e^{i(\phi-v)} \end{bmatrix}$$

Using the Pauli matrix algebra developed earlier we may write

$$T^* = e^{-\tau_1\sigma_1} e^{-\theta_1\sigma_2}$$

$$T^{*T} = e^{-\tau_1\sigma_1} e^{\theta_1\sigma_2}$$

$$D = e^{\frac{1}{2}\nu\sigma_3} \begin{bmatrix} m & 0 \\ 0 & m\tan^2\gamma \end{bmatrix} e^{\frac{1}{2}\nu\sigma_3}$$

Thus

$$[S] = e^{-\tau_1\sigma_1} e^{-\theta_1\sigma_2} e^{\frac{1}{2}\nu\sigma_3} \begin{bmatrix} m & 0 \\ 0 & m\tan^2\gamma \end{bmatrix} e^{\frac{1}{2}\nu\sigma_3} e^{-\tau_1\sigma_1} e^{\theta_1\sigma_2}$$

This form is used when developing relationships between the coherent theory and Stokes matrix. This is the next logical step in the theory but is beyond the scope of these lectures. For those interested, further details may be obtained in references 5 and 15. (See also appendix 1).

CLASSIFICATION OF RADAR TARGETS

In the previous section the target vector V was developed which has five parameters and presents a full polarimetric descriptor of the target. In this chapter some time will be spent investigating the form of V for some simple radar scatterers and outlining a basic scheme for classifying radar targets according to their polarimetric properties. Note that we are still considering fixed aspect and frequency data and coherent scattering in the far field of the target. The problems of scintillating targets and change of aspect and frequency will be briefly considered later.

Let us define the polarisation spectrum of a target with scattering matrix [S] by the copolar and crosspolar RCS both given as a function of polarisation. Shown in Figures 18 and 19 is the spectrum of a simple flat plate or trihedral target. On these plots θ is mapped from $0 \rightarrow 180^{\circ}$ versus τ the ellipticity ($-45^{\circ} < \tau < 45^{\circ}$). Notice that the copolar plot shows a maximum RCS for all linear polarisations and a null for left and right circular. The crosspolar plot shows a maximum for circular and a corresponding null for all linear. These properties may be summarised by writing the target vector

$$V = (0^{\circ}, 0^{\circ}, 0^{\circ}, 45^{\circ}, m) \qquad m = \frac{4\pi A^2}{\lambda^2}$$

A - area of plate
λ - illuminating wavelength

When mapped onto the Poincaré sphere the polarisation fork for this target is as shown in Figure 20. The fork has maximum angle separation (γ) which is indicative of the copolar nulls being orthogonal. Notice that because the eigenpolarisations include the whole equator this target is rotation invariant ie there is no change in the fork if the target is rotated about the radar line of sight.

Compare these results with those for the dihedral reflector as shown in Figures 20-23. The dihedral is assumed to have its seam parallel to the radar vertical and thus has two nulls at $\pm 45^{\circ}$ linear and maxima for all polarisations with $\theta = 0^{\circ}$ or 90° and $-45^{\circ} < \tau < 45^{\circ}$. The target vector is

$$V = (0^{\circ}, 0^{\circ}, 90^{\circ}, 45^{\circ}, m)$$

The polarisation fork is similar to that of the trihedral except for a 90° rotation about the eigenpolarisation axis. This is the skip angle and relates to the number of bounces the reflected signal undergoes. In general however the skip angle is the phase angle between eigenvalues and so does not lend itself to such simple physical interpretation. Note that because of the form of polarisation fork the dihedral is rotation dependent and for a general rotation angle ψ the target vector becomes

$$V = (2\psi, 0, 90^\circ, 45^\circ, m)$$

ie the fork is rotated in the equatorial plane by $2\psi^\circ$.

The wire grid target considered earlier has the polarisation fork shown in Figure 24. In this case

$$V = (2\psi, 0^\circ, 0^\circ, 0^\circ, m)$$

Notice that the copolar nulls are now coincident and the fork has collapsed into a straight line. For the grid at an arbitrary angle ψ it matrix is

$$[S] = \begin{bmatrix} \cos^2\psi & \frac{1}{2}\sin 2\psi \\ \frac{1}{2}\sin 2\psi & \sin^2\psi \end{bmatrix}$$

and as such the fork rotates by 2ψ in the equatorial plane.

These are all examples of an important class of symmetric targets. The general form of [S] in (x,y) for a symmetric target is

$$S = \begin{bmatrix} e^{2iv} & 0 \\ 0 & \tan^2\gamma \, e^{-2iv} \end{bmatrix} \qquad \text{where } (x,y) \text{ is a linear base}$$

All targets having an axis of roll symmetry are symmetric at any aspect angle, eg cones, cylinders, ellipsoids and combinations of these. Other targets like corner reflectors may have a plane of symmetry through a line of sight direction and thus be symmetrical targets. All these scatterers have a target vector

$$V = (\theta, 0^\circ, v, \gamma, m) \qquad \text{ie } \tau_m = 0^\circ$$

This may be illustrated by virtue of Figure 25 which shows a general symmetric radar target and a proposed maximum polarisation $\hat{P}_{max}$. Because of the symmetry of the object there must then be another maximum polarisation $\hat{P}_{max\,z}$ obtained from $\hat{P}_{max}$ by reflection in the plane of symmetry. However theory states that the eigenpolarisations are orthogonal and so $\hat{P}_{max}$ must be either parallel or perpendicular to the symmetry axis. For either choice $\tau_m = 0$ and the eigenpolarisations are linear.

Symmetric targets are characterised by their eigenpolarisations always lying in the equatorial plane and are of special interest in radar scattering studies since many of the most common radar targets show these features.

The class of non-symmetric targets are classified by $\tau_m \neq 0$. The maximum value that τ_m may achieve is $\pm 45^\circ$ and it does this for a helix target which has the matrix

$$S = \begin{bmatrix} 1 & J \\ J & -1 \end{bmatrix}_{\text{left screw}} \quad \text{or} \quad \begin{bmatrix} 1 & -J \\ -J & -1 \end{bmatrix}_{\text{right screw}}$$

The corresponding forks are shown in Figures 26 and 27 and for these targets

$$V = (0^\circ, \pm 90^\circ, 0^\circ, m)$$

Because these represent an extreme in target asymmetry the angle τ_m is termed the helicity angle.

A class of nonsymmetric targets which has special significance in the decomposition theorems of Huynen are n-targets (nonsymmetric noise) defined by

$$\tau_m = \pm 45^\circ$$

Thus [S] has the general form

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

Note that the dihedral target may be considered as a special case of an n-target since

$$[S] = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$$

So far only single, isolated targets have been considered but in reality radar targets will be composed of many scattering centres each with different polarimetric properties and target/radar motion will result in scintillation of the target echo. In the fixed polarisation scalar theory these fluctuations have received much attention but little is known of the dynamic properties of the target vector V and their relation to changes of radar frequency and target aspect. Certain interesting trends may be pointed out however.

If the target has a few dominant scatterers with different polarimetric properties then the loci of null polarisations for charge of aspect or frequency might be expected to cover large areas of the Poincaré sphere but if on the other hand it consists of reflectors with similar polarimetric properties or many scatterers distributed within a narrow dynamic range, then clustering of the null polarisations might be expected. This has been verified for rain clutter[7] and provides the basis for polarisation filtering whereby controlled suppression may be had for limited areas of polarisation space. The frequency dependence of V is an important area of study and has been covered by Boerner[8]. This work promises to combine polarimetric information with spatial range information available from broadband systems and as such provide a powerful theory for electromagnetic inverse scattering.

The incorporation of full polarimetric information with radar systems has many widespread applications. The efficient suppression of rain clutter is but one example and in general any target whose nulls show clustering in polarisation space may be enhanced or suppressed using these techniques. The important point to note is that SM methods allow the choice of effective transmit/receive polarisation to be made in the signal processing and real time adaptation of these is then limited only by the speed of processing available.

The techniques have application in improving signal to clutter ratio, ECCM, multipath reduction and radar target identification studies. In particular the future will undoubtedly witness the incorporation of polarimetric techniques with other features of the target's signature like doppler or broadband interrogation for advanced remote sensing applications.

APPENDIX 1. DERIVATION OF STOKES REFLECTION MATRIX

The coherent theory may be related to the Stokes calculus by considering again the properties of the coherency matrix [J].

The general polarisation vector (spinor) may be written

$$\underset{\sim}{E} = \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

then

$$\underset{\sim}{E}^{+} = \text{adjoint} = \text{conjugate transpose} = (E_x{}^*, E_y{}^*)$$

The products of $\underset{\sim}{E}$ and $\underset{\sim}{E}^{+}$ are

$$\underset{\sim}{E}^{+}\underset{\sim}{E} = I = \text{intensity} = \text{scalar}$$

$$\underset{\sim}{E}\underset{\sim}{E}^{+} = J = \text{coherency matrix}$$

If the target has x,y corrected scattering matrix [T] then

$$\underset{\sim}{E}_s = [T]\underset{\sim}{E} \qquad \underset{\sim}{E}_s \text{ - scattered polarisation vector}$$

Under these circumstances the transformed coherency matrix is

$$J_s = \underset{\sim}{E}_s\underset{\sim}{E}_s{}^{+}$$

$$= [T]\ J\ [T]^{+}$$

From the properties of the Pauli matrices

$$\underline{\sigma} = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$$

$$J_s = \tfrac{1}{2} \sum_{\mu} S_{\mu s}\, \sigma_{\mu} \qquad S_{\mu s} = T_r(J_s\, \sigma_{\mu}) \qquad \mu = 0,1,2,3$$

$S_{\mu s}$ are the coefficients of expansion for the transformed coherency matrix. Writing this in compact form and using the cyclic property of the Trace operator

$$S_s = \tfrac{1}{2} T_r(J_s \underline{\sigma})$$

$$= \tfrac{1}{2} T_r([T]\ J\ [T]^+ \underline{\sigma})$$

$$= \tfrac{1}{2} T_r(J[T]^+ \underline{\sigma}[T])$$

$$= \tfrac{1}{2} T_r(J\ \underline{\sigma}_s)$$

where

$$\underline{\sigma}_s = [T]^+ \underline{\sigma}[T]$$

The coherency matrix may be written

$$J = \tfrac{1}{2} \sum_v T_r(J\ \sigma_v)\sigma_v \qquad v = 0,1,2,3$$

and

$$\underline{\sigma}_s = \tfrac{1}{2} \sum_v T_r(\underline{\sigma}_s\ \sigma_v)\sigma_v$$

Thus

$$S_s = \tfrac{1}{4} \sum_v T_r(\underline{\sigma}_s\ \sigma_v) T_r(J\ \sigma_v) \qquad \text{since } (\sigma_v)^2 = \sigma_0$$

$$= \tfrac{1}{2} \sum_v M_{\mu v}\ S_v$$

We have now found a matrix M whose element $M_{\mu v}$ relates the vth Stokes parameter of input wave to the μth Stokes parameter of reflected wave where

$$M_{\mu v} = \tfrac{1}{2}\ T_r(\sigma_{\mu s}\ \sigma_v)$$

$$\sigma_{\mu s} = [T]^{+} \sigma_{\mu} [T]$$

This relationship gives a method of determining the Stokes reflection matrix M from any given target matrix [T]

$$\underline{g}_s = [M] \underline{g}$$

$$[M] = \tfrac{1}{2} T_r(\underline{\sigma}_s \underline{\sigma})$$

where

$$\underline{\sigma}_s \underline{\sigma} = \begin{bmatrix} \sigma_{0s} \\ \sigma_{1s} \\ \sigma_{2s} \\ \sigma_{3s} \end{bmatrix} (\sigma_0 \; \sigma_1 \; \sigma_2 \; \sigma_3)$$

$$= \begin{bmatrix} \sigma_{0s}\sigma_0 & \sigma_{0s}\sigma_1 & \sigma_{0s}\sigma_2 & \sigma_{0s}\sigma_3 \\ \sigma_{1s}\sigma_0 & \sigma_{1s}\sigma_1 & \sigma_{1s}\sigma_2 & \sigma_{1s}\sigma_3 \\ \sigma_{2s}\sigma_0 & \sigma_{2s}\sigma_1 & \sigma_{2s}\sigma_2 & \sigma_{2s}\sigma_3 \\ \sigma_{3s}\sigma_0 & \sigma_{3s}\sigma_1 & \sigma_{3s}\sigma_2 & \sigma_{3s}\sigma_3 \end{bmatrix}$$

which after taking the trace gives

$$[M] = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & [illegible] \end{bmatrix}$$

This expression for [M] is in the natural ordering and in order to convert to traditional

$$[M]_{trad} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} [M]_{natural} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Thus, using the same notation as above

$$[M]_{trad} = \begin{bmatrix} m_{11} & m_{14} & m_{12} & m_{13} \\ m_{41} & m_{44} & m_{42} & m_{43} \\ m_{21} & m_{24} & m_{22} & m_{23} \\ m_{31} & m_{34} & m_{32} & m_{33} \end{bmatrix}$$

Thus

$$\underline{g}_s = [M]\underline{g}$$

Before working out an explicit form for $[M]$ care must again be taken to ensure that the incident and reflected waves are referenced to the same co-ordinate system. $[M]$ has already been corrected for x,y plane co-ordinates through $[T]$ but in order that the conjugate nature of reflected wave co-ordinates be taken into account.

$$\underline{g}_s' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \underline{g}_s = [Q]\,\underline{g}_s$$

Thus

$$\underline{g}_s' = [Q][M]\underline{g}_s$$

If we then define

$$[M]' = [Q][M]$$

then we may use $[M]'$ to relate an incident wave $\hat{P}$ to a reflected wave where both are defined in terms of the transmitter co-ordinates and write the expression for received power P as a dot product of Stokes vectors

$$P = [M]' \, \underline{g}(\hat{P}) \quad g(\hat{Q}) \qquad \underline{g}(\hat{P}) \text{ - Stokes vector for } \hat{P}$$

$$\underline{g}(\hat{Q}) \text{ - Stokes vector for } \hat{Q}$$

When multiplied out

$$[M]' = \begin{bmatrix} T_r(\sigma_{0s}\sigma_0) & T_r(\sigma_{0s}\sigma_3) & T_r(\sigma_{0s}\sigma_1) & T_r(\sigma_{0s}\sigma_2) \\ T_r(\sigma_{3s}\sigma_0) & T_r(\sigma_{3s}\sigma_3) & T_r(\sigma_{3s}\sigma_1) & T_r(\sigma_{3s}\sigma_2) \\ T_r(\sigma_{1s}\sigma_0) & T_r(\sigma_{1s}\sigma_3) & T_r(\sigma_{1s}\sigma_1) & T_r(\sigma_{1s}\sigma_2) \\ -T_r(\sigma_{2s}\sigma_0) & -T_r(\sigma_{2s}\sigma_3) & -T_r(\sigma_{2s}\sigma_1) & -Tr(\sigma_{2s}\sigma_2) \end{bmatrix}$$

If we put

$$[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad aa^* = a^2 \text{ etc}$$

$$\mathrm{Re}(a) = \frac{a+a^*}{2} \qquad \mathrm{Im}(a) = \frac{a-a^*}{2}$$

then $[M]'$ becomes

$$\frac{1}{2}\begin{bmatrix} \frac{1}{2}(a^2+b^2+c^2+d^2) & \frac{1}{2}(a^2-b^2+c^2-d^2) & \mathrm{Re}(a^*b + c^*d) & i\mathrm{Im}(a^*b+c^*d) \\ \frac{1}{2}(a^2+b^2-c^2-d^2) & \frac{1}{2}(a^2-b^2-c^2+d^2) & \mathrm{Re}(a^*b - c^*d) & i\mathrm{Im}(a^*b-c^*d) \\ \mathrm{Re}(a^*c + b^*d) & \mathrm{Re}(a^*c - b^*d) & \mathrm{Re}(a^*d + b^*c) & i\mathrm{Im}(a^*d-b^*c) \\ i\mathrm{Im}(a^*c+b^*d) & i\mathrm{Im}(a^*c-b^*d) & i\mathrm{Im}(a^*d+b^*c) & \mathrm{Re}(b^*c - a^*d) \end{bmatrix}$$

Notice that for symmetric $[T]$, $[M]'$ is also symmetric with nine independent elements. For the general bistatic case however there will be sixteen elements although symmetry in the scatterer may reduce this number.

For partially polarised waves, averages are taken of the individual Stokes parameters and so one may define an average Stokes reflection matrix by

$$[M]'_{av} = <[M]> \qquad \text{where} \quad m_{iJav} = <m_{iJ}>$$

Huynen's[5] decomposition is based on the fact that this averaged matrix may be decomposed into a component with corresponding coherent matrix [T] satisfying the condition for physical realisability of Stokes vectors, plus a remainder matrix which may be decomposed itself into the sum of two n-targets.

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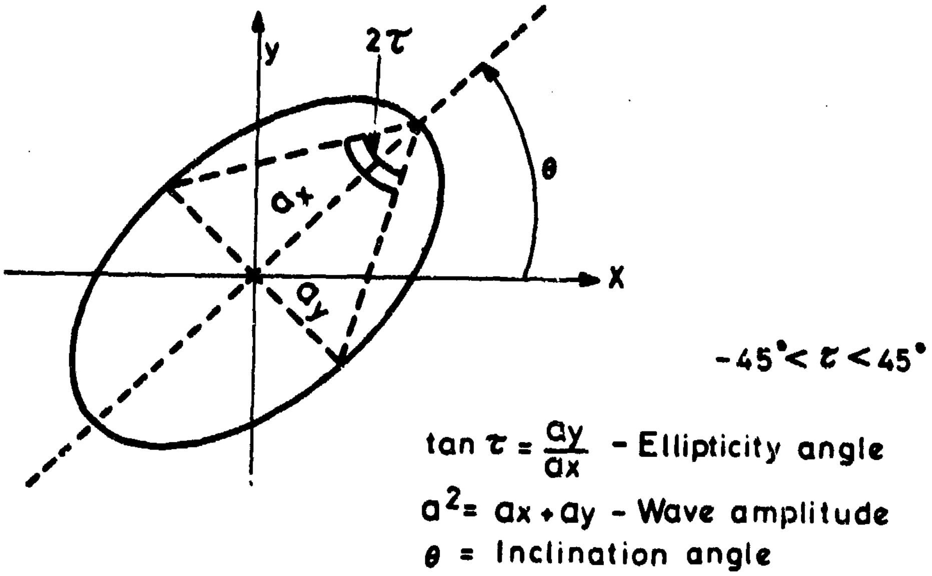


FIG.1 POLARISATION ELLIPSE

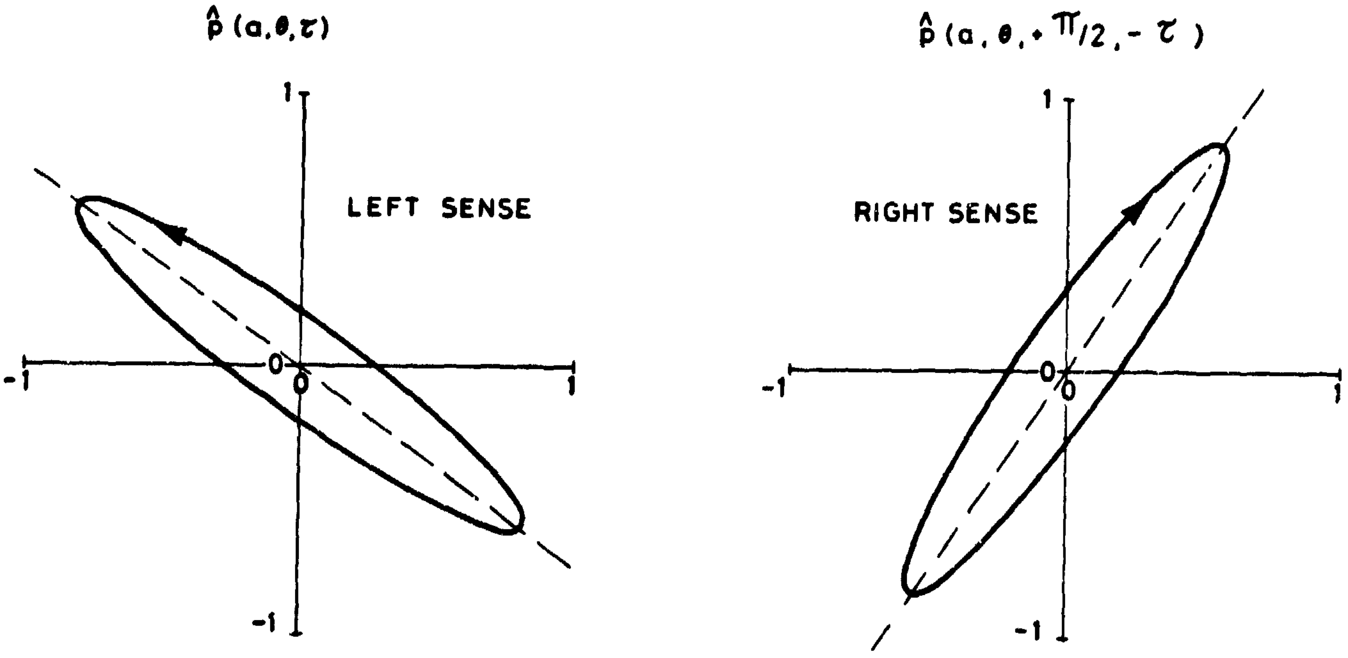
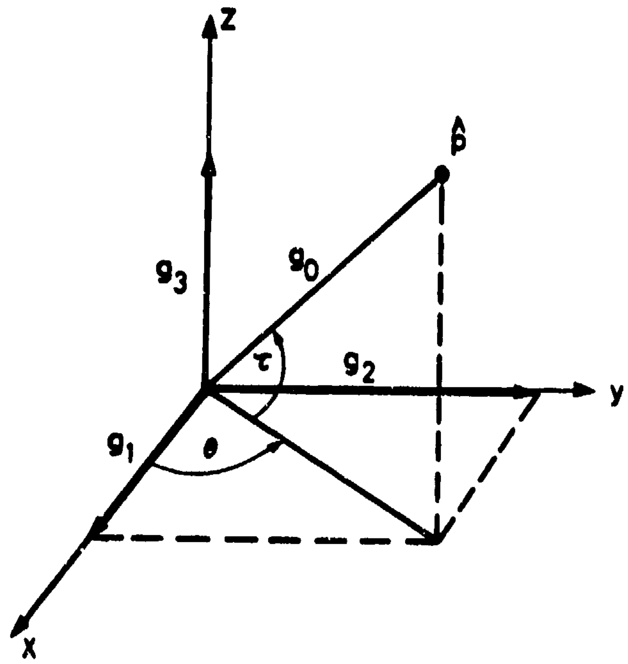


FIG. 2 ORTHOGONAL POLARISATIONS



$\underline{g} = (g_0\ g_1\ g_2\ g_3)$ $\qquad$ $g_0^2 = g_1^2 + g_2^2 + g_3^2$

(1, 1, 0, 0) = Horizontal
(1,-1, 0, 0) = Vertical
(1, 0, 1, 0) = +45° Linear
(1, 0, 0, 1) = Left Circular

FIG. 3 STOKES PARAMETERS AND POLARISATION SPACE

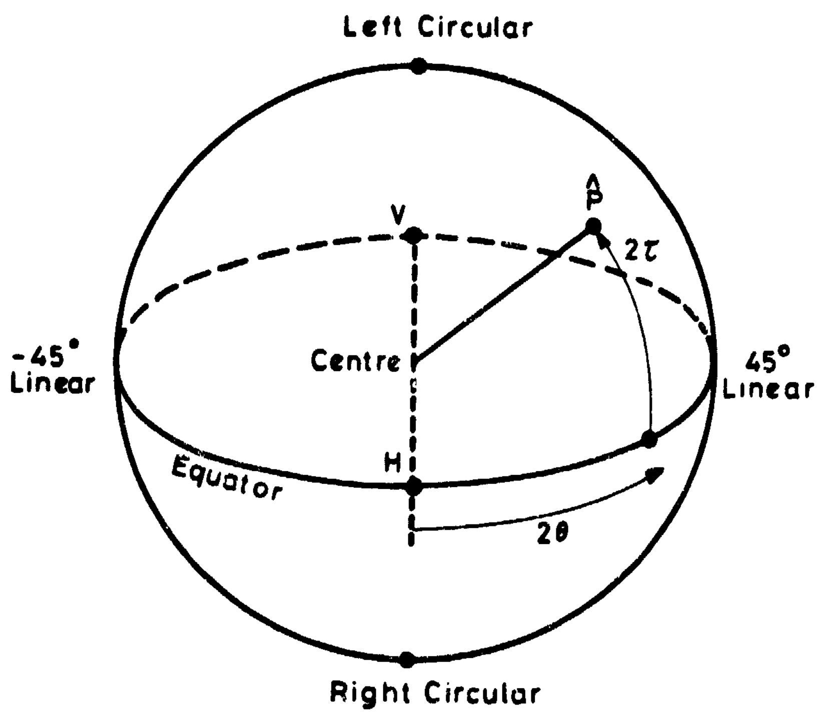
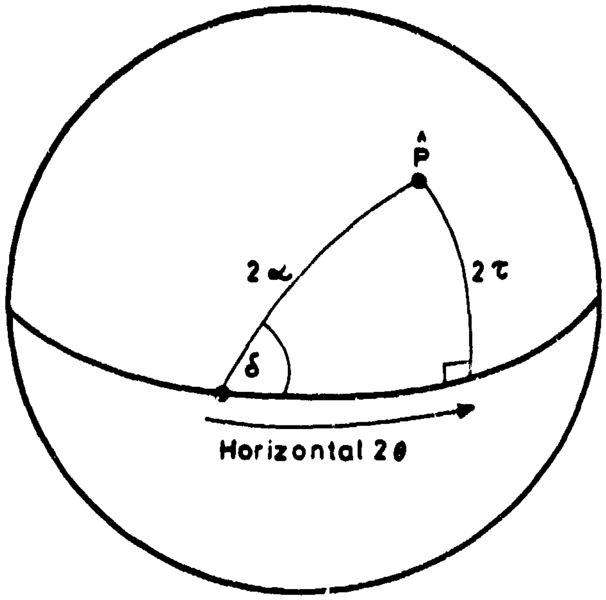


FIG. 4 POINCARE SPHERE



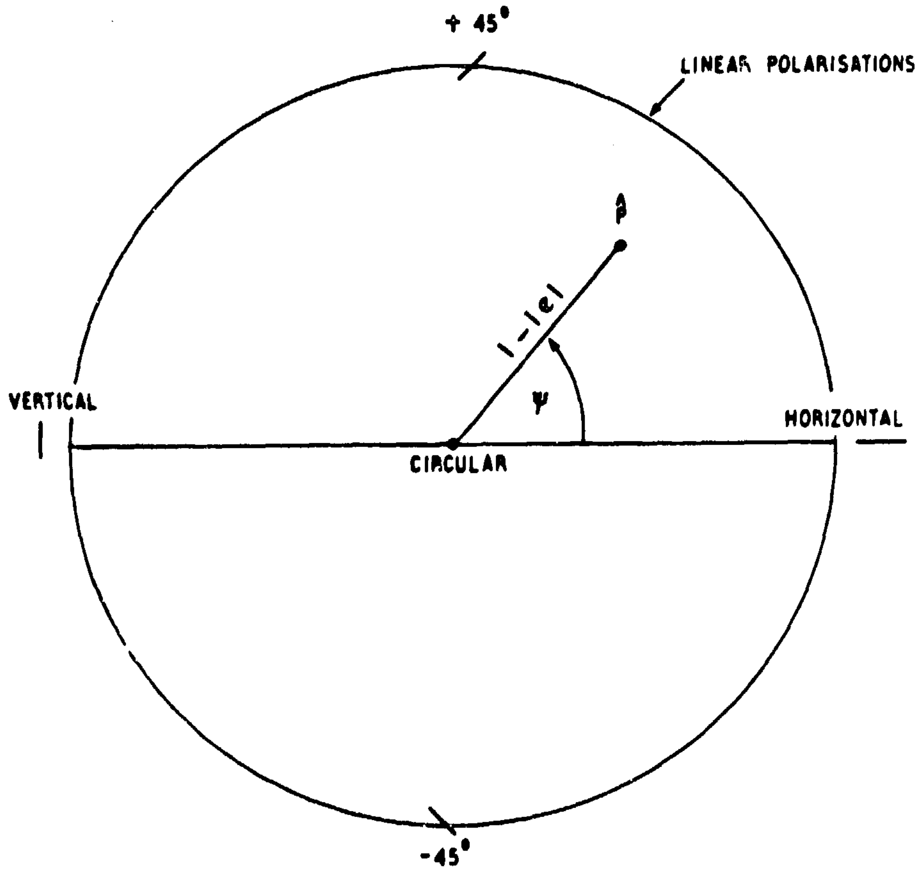
δ = Phase of polarisation ratio

$\tan\alpha$ = Amplitude of polarisation ratio

θ = Inclination angle of polarisation ellipse

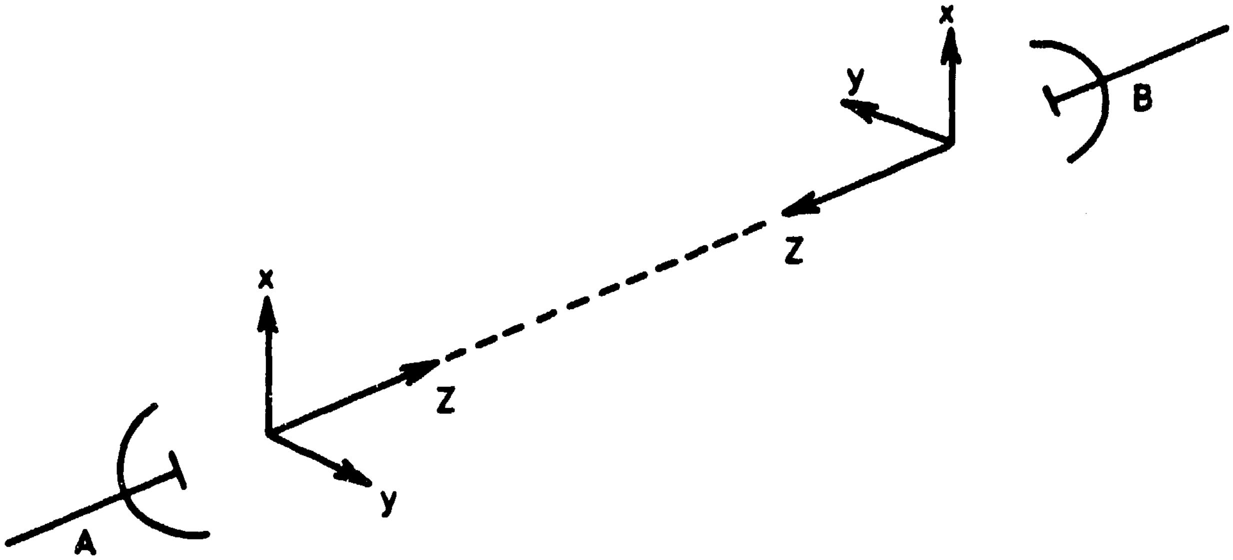
τ Ellipticity angle

FIG. 5 DÉSCHAMPS SPHERE



RADIUS FROM $C-\hat{p} = 1-|e|$ $|e|$ IS ECCENTRICITY OF POLARISATION ELLIPSE ($0 \leq |e| \leq 1$)

ANGLE $\psi = 2\theta$ θ IS THE ORIENTATION ANGLE OF POLARISATION ELLIPSE ($-\pi/2 \leq \theta \leq \pi/2$)



$$\begin{bmatrix} X \\ y \end{bmatrix}_A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} X \\ y \end{bmatrix}_B$$

FIG. 7 ANTENNA CO-ORDINATE SYSTEMS

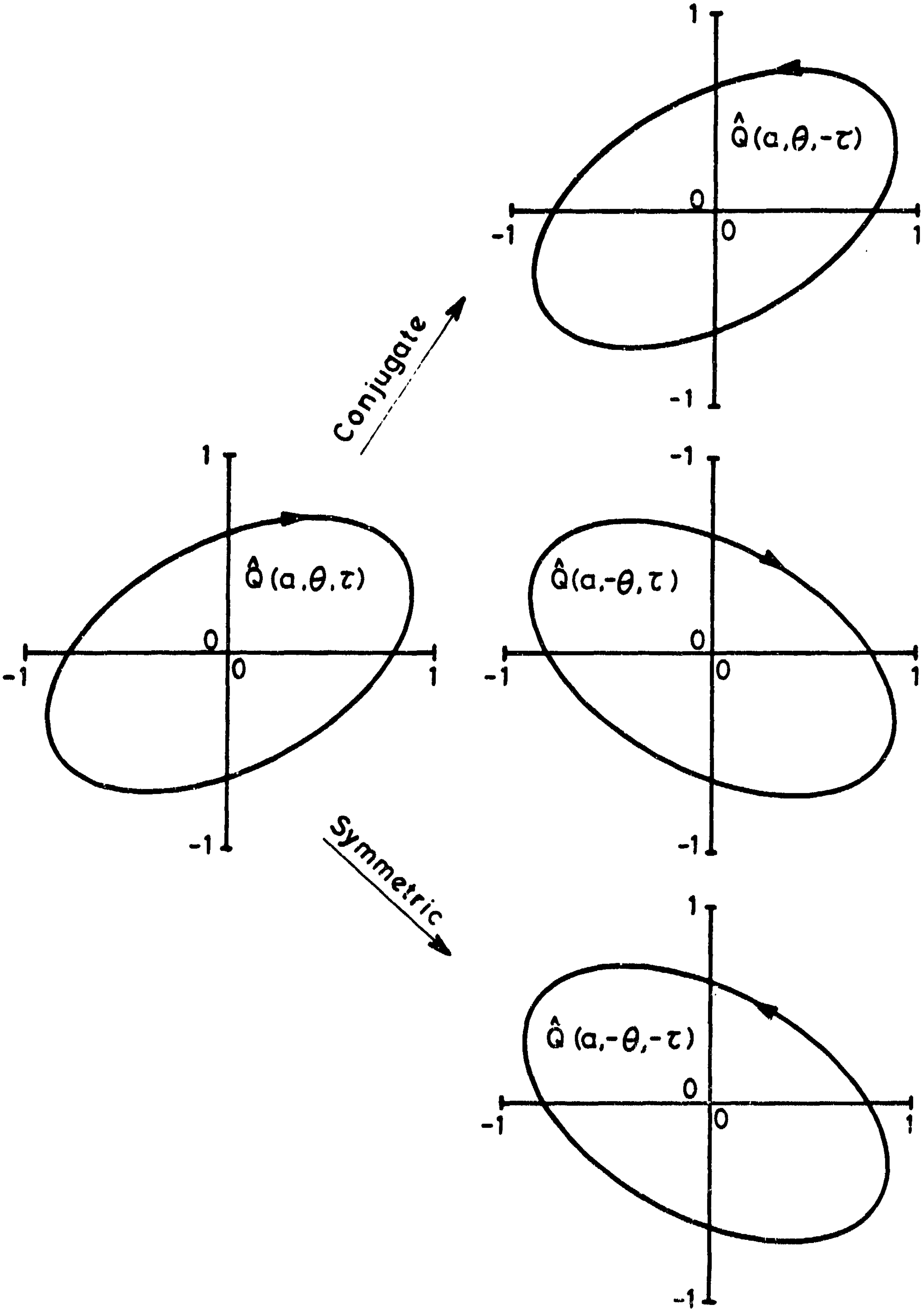
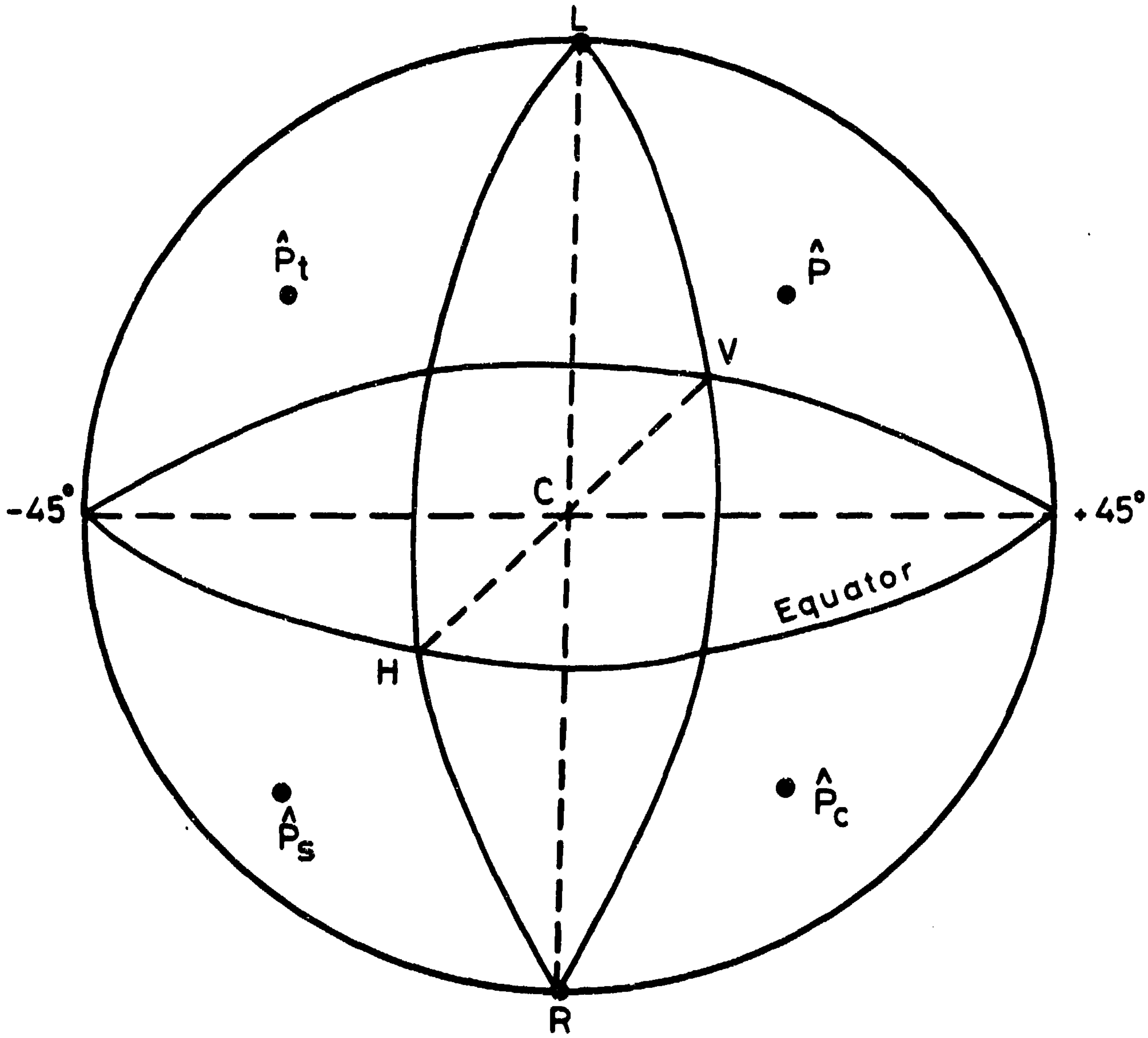


FIG. 8 SPECIAL POLARISATIONS



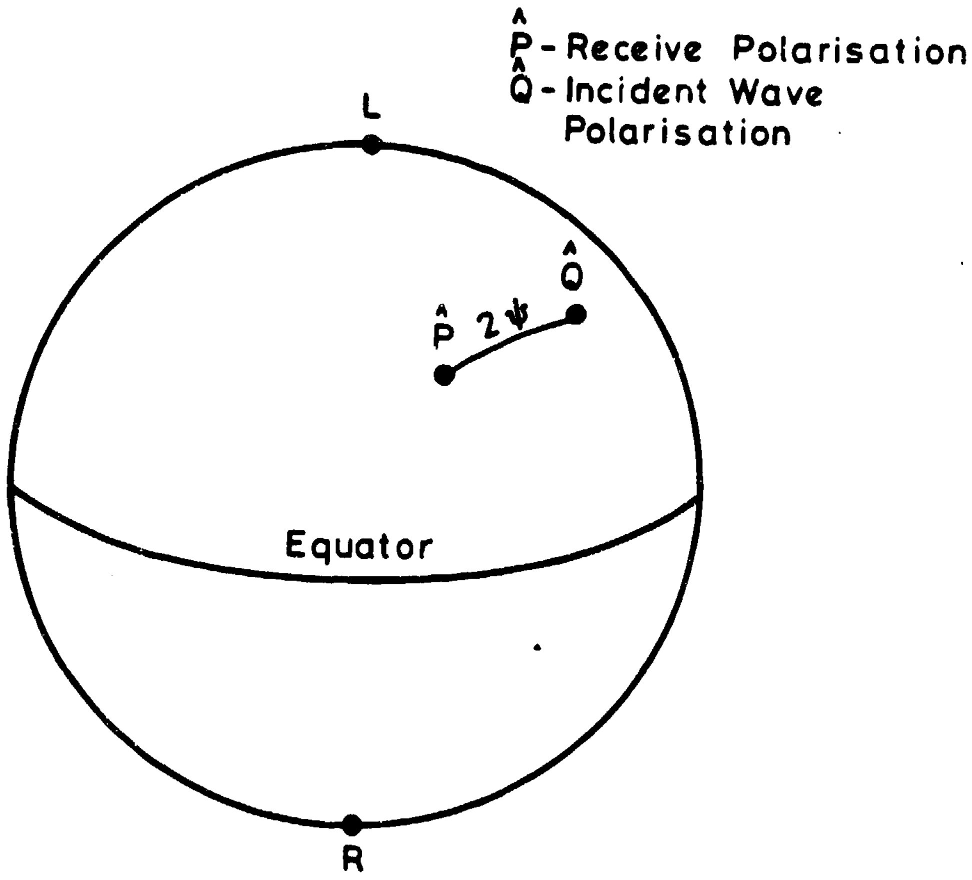
$\hat{P}$ - Parent

$\hat{P}_c$ - Conjugate - REFLECTION IN EQUATORIAL PLANE

$\hat{P}_s$ - Symmetric - REFLECTION IN H,V DIAMETER

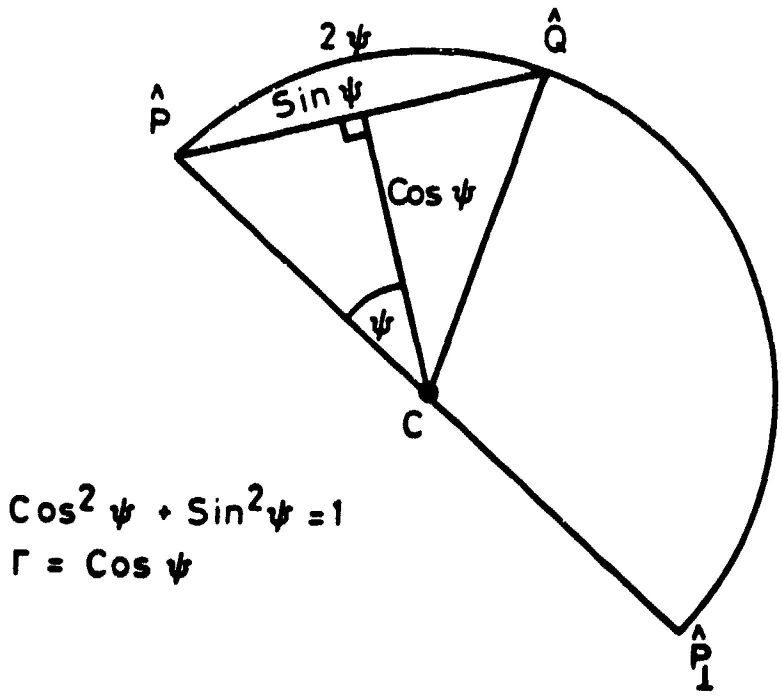
$\hat{P}_t$ - Transverse - REFLECTION IN HLVR PLANE

FIG.9 $\hat{P}$ AND ITS RELATED POLARISATIONS ON THE POINCARE SPHERE



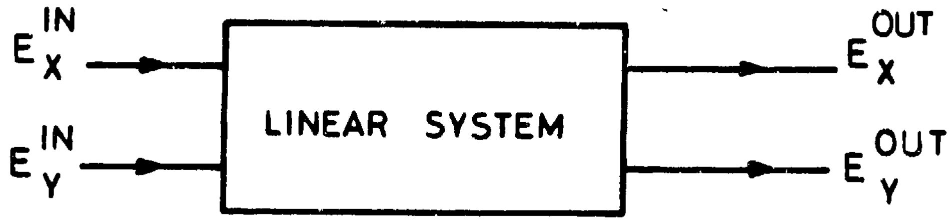
$\hat{P}$ - Receive Polarisation
$\hat{Q}$ - Incident Wave Polarisation
L
$\hat{Q}$
$\hat{P}$
2ψ
Equator
R

FIG. 10



2ψ
$\hat{Q}$
$\hat{P}$
$\sin\psi$
$\cos\psi$
ψ
C
$\cos^2\psi + \sin^2\psi = 1$
$\Gamma = \cos\psi$
$\hat{P}_\perp$

FIG. 11

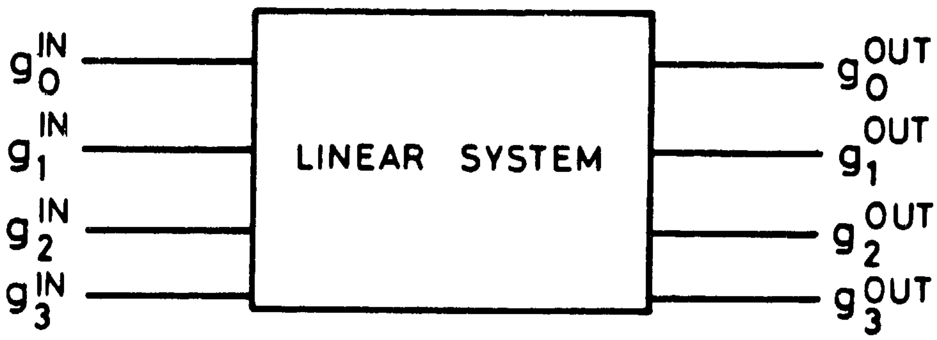


$$E_X^{OUT} = S_{11} E_X^{IN} + S_{12} E_Y^{IN}$$

$$E_Y^{OUT} = S_{21} E_X^{IN} + S_{22} E_Y^{IN}$$

$$\begin{bmatrix} E_X \\ E_Y \end{bmatrix}_{OUT} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} E_X \\ E_Y \end{bmatrix}_{IN} \qquad S_{ij} = |S_{ij}| e^{j\theta_{ij}}$$

COHERENT SCATTERING MATRIX



$$\begin{bmatrix} g_0^{OUT} \\ g_1^{OUT} \\ g_2^{OUT} \\ g_3^{OUT} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} g_0^{IN} \\ g_1^{IN} \\ g_2^{IN} \\ g_3^{IN} \end{bmatrix}$$

$$m_{ij} = |m_{ij}|$$

MUELLER MATRIX

FIG. 12

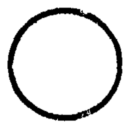
TARGET	$[S]_{(HV)}$	$[M]_{(H,V)}$
SPHERE 	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
DIHEDRAL @ 0° 	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
DIHEDRAL @ ψ° 	$\begin{bmatrix} \text{Cos}\,2\psi & \text{Sin}\,2\psi \\ \text{Sin}\,2\psi & -\text{Cos}\,2\psi \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \text{Cos}\,4\psi & \text{Sin}\,4\psi & 0 \\ 0 & \text{Sin}\,4\psi & \text{Cos}\,4\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
LINEAR TARGET @ 0°	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
LINEAR TARGET @ ψ	$\begin{bmatrix} \text{Cos}^2\psi & \frac{1}{2}\text{Sin}\,2\psi \\ \frac{1}{2}\text{Sin}\,2\psi & \text{Sin}^2\psi \end{bmatrix}$	$\begin{bmatrix} 1 & \text{Cos}\,2\psi & \text{Sin}\,2\psi & 0 \\ \text{Cos}\,2\psi & \text{Cos}^2 2\psi & \frac{1}{2}\text{Sin}\,4\psi & 0 \\ \text{Sin}\,2\psi & \frac{1}{2}\text{Sin}\,4\psi & \text{Cos}^2 2\psi & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
LEFT HAND HELIX 	$\begin{bmatrix} 1 & J \\ J & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$
RIGHT HAND HELIX 	$\begin{bmatrix} 1 & -J \\ -J & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

FIG.13

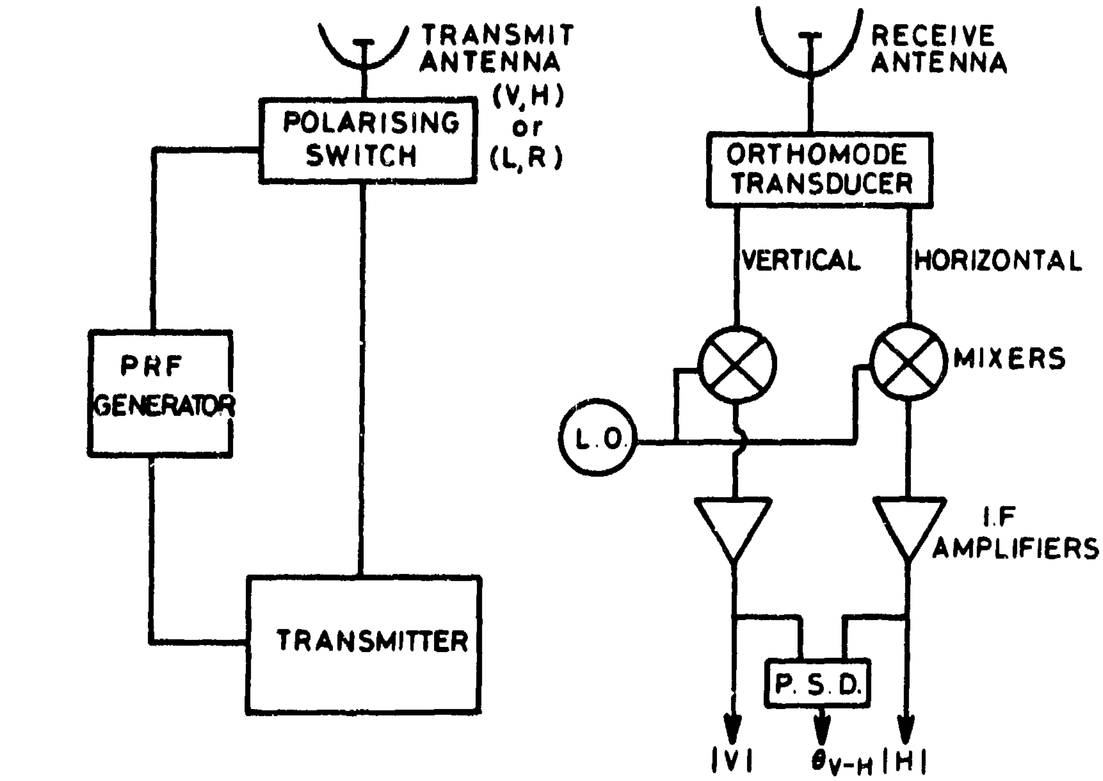


FIG. 14 SCHEMATIC DIAGRAM OF PULSED RADAR SYSTEM SUITABLE FOR MEASUREMENT OF THE RELATIVE SCATTERING MATRIX

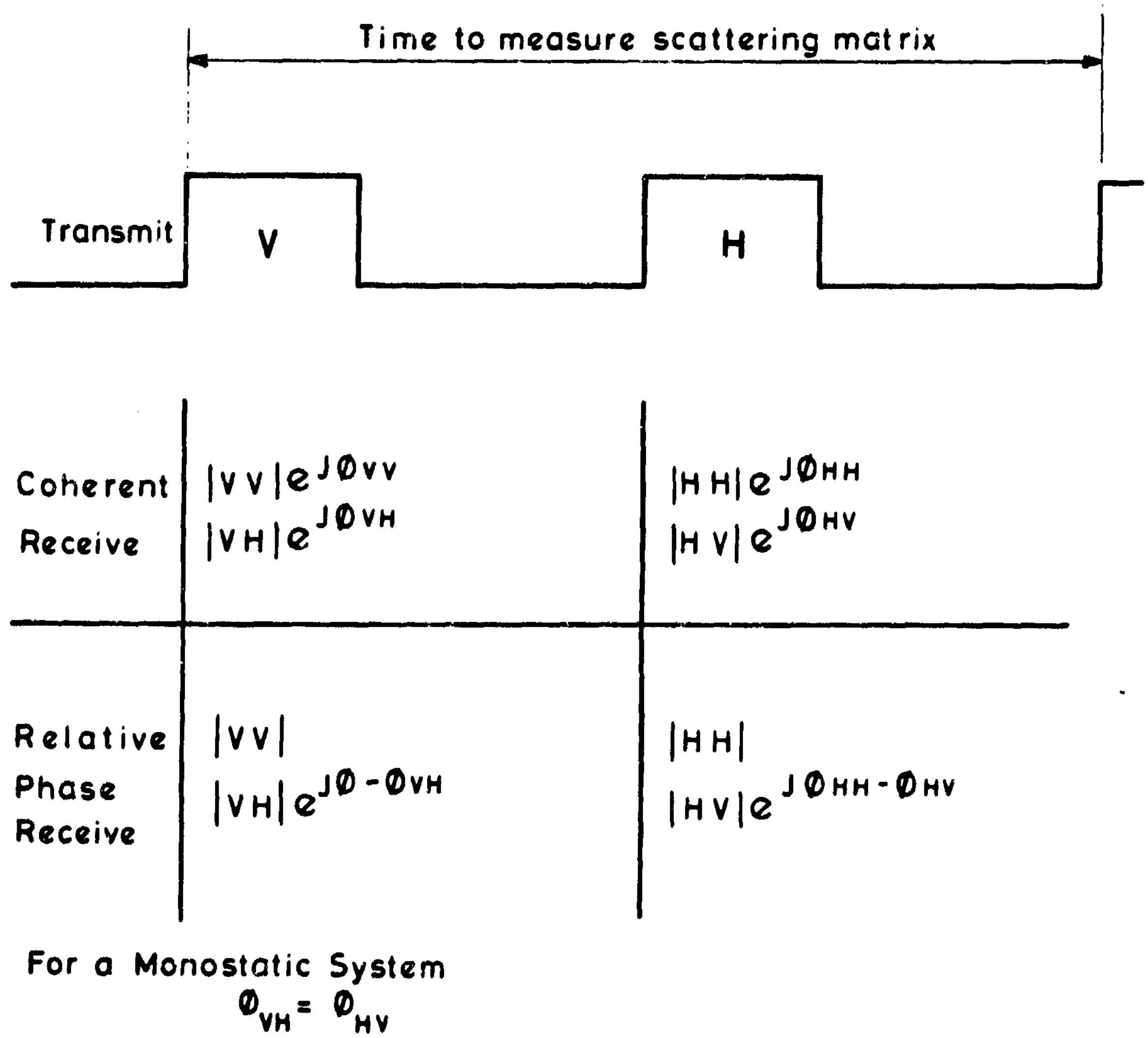
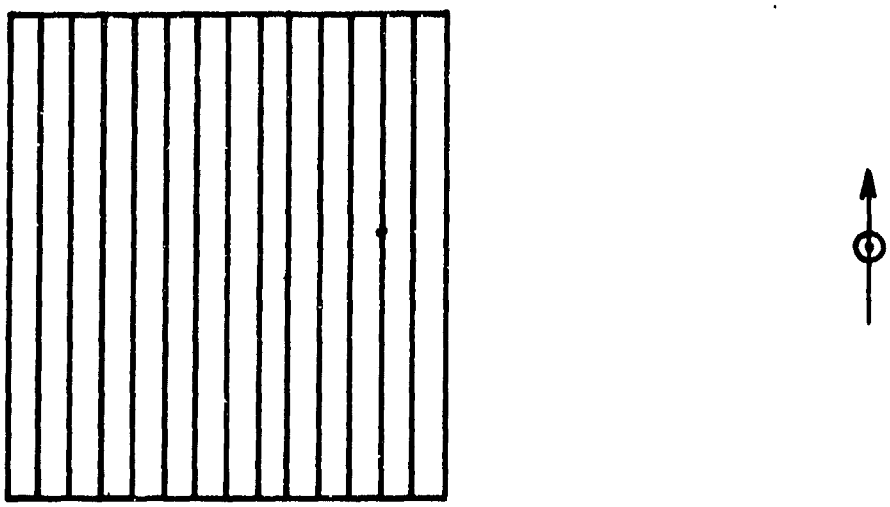


FIG. 15



Vertical is an Eigenpolarisation

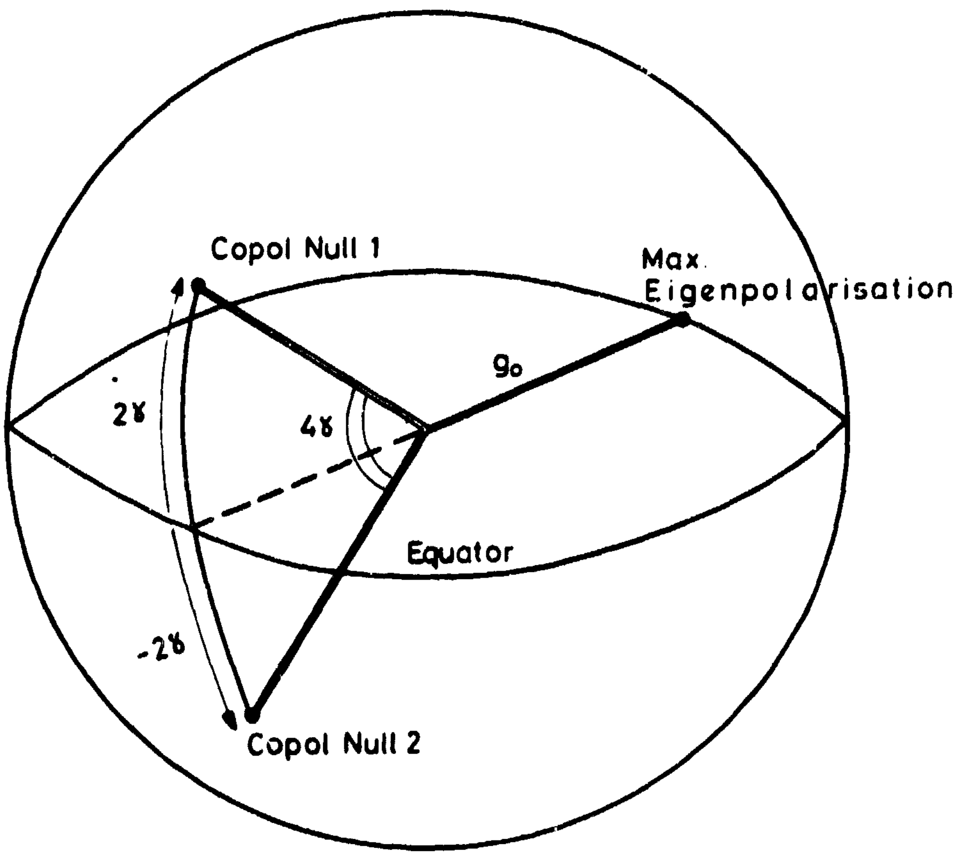
Horizontal is both an Eigenpolarisation and a Copolar Null

$$\begin{bmatrix} h \\ v \end{bmatrix}_{Out} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h \\ v \end{bmatrix}_{In}$$

For grid at Angle ϕ°

$$\begin{bmatrix} h \\ v \end{bmatrix}_{Out} = \begin{bmatrix} \sin^2\phi & \frac{1}{2}\sin 2\phi \\ \frac{1}{2}\sin 2\phi & \cos^2\phi \end{bmatrix} \begin{bmatrix} h \\ v \end{bmatrix}_{In}$$

FIG.16 SCATTERING FROM WIRE GRID



The fork lies in a plane defined by $(\theta_{MAX}, \tau_{MAX}, \gamma)$ representing rotations about Z,Y,X axes respectively

COPOLAR RCS AS FN OF POLARISATION

FIG.18

TRIHEDRAL

$[S] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

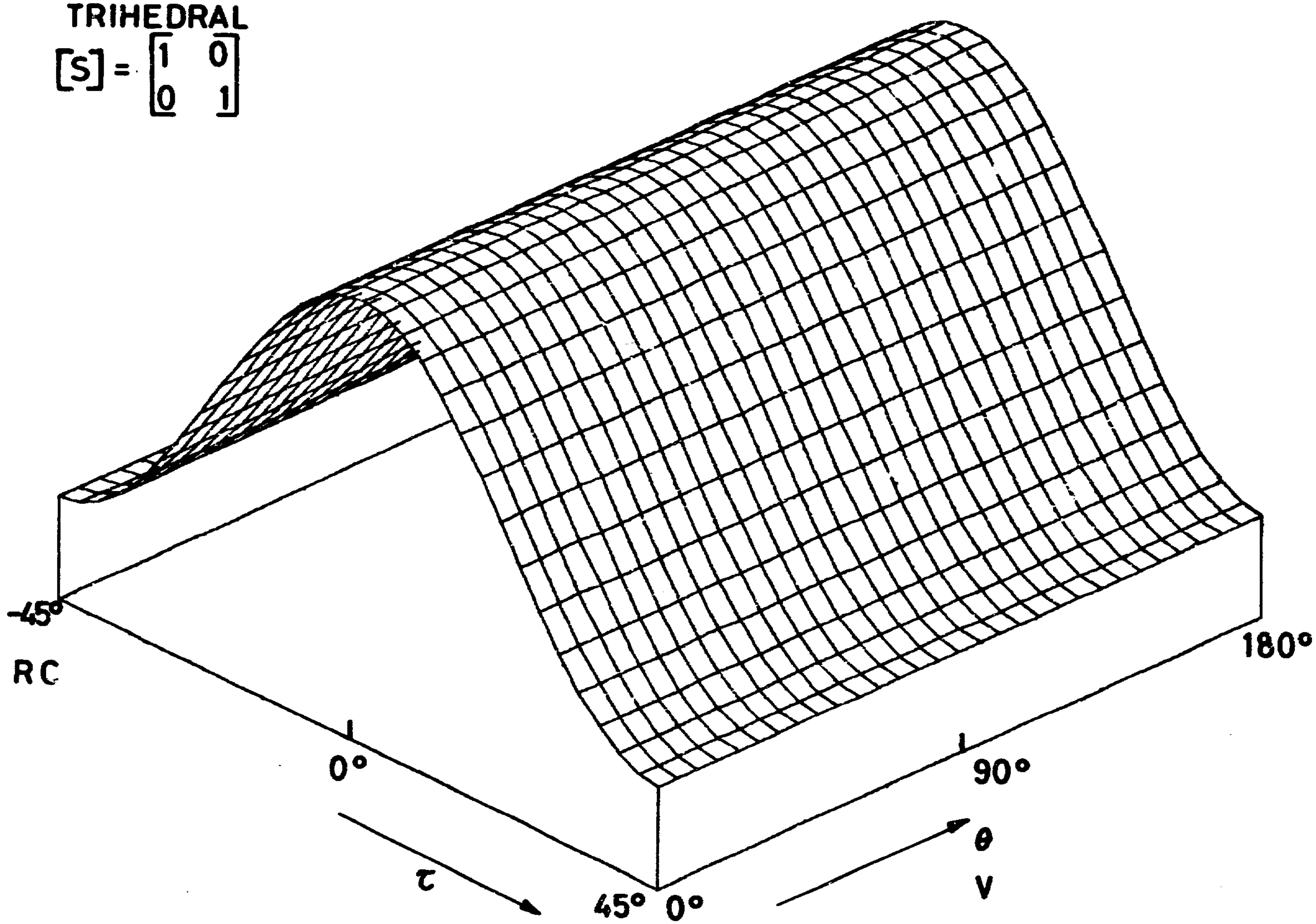


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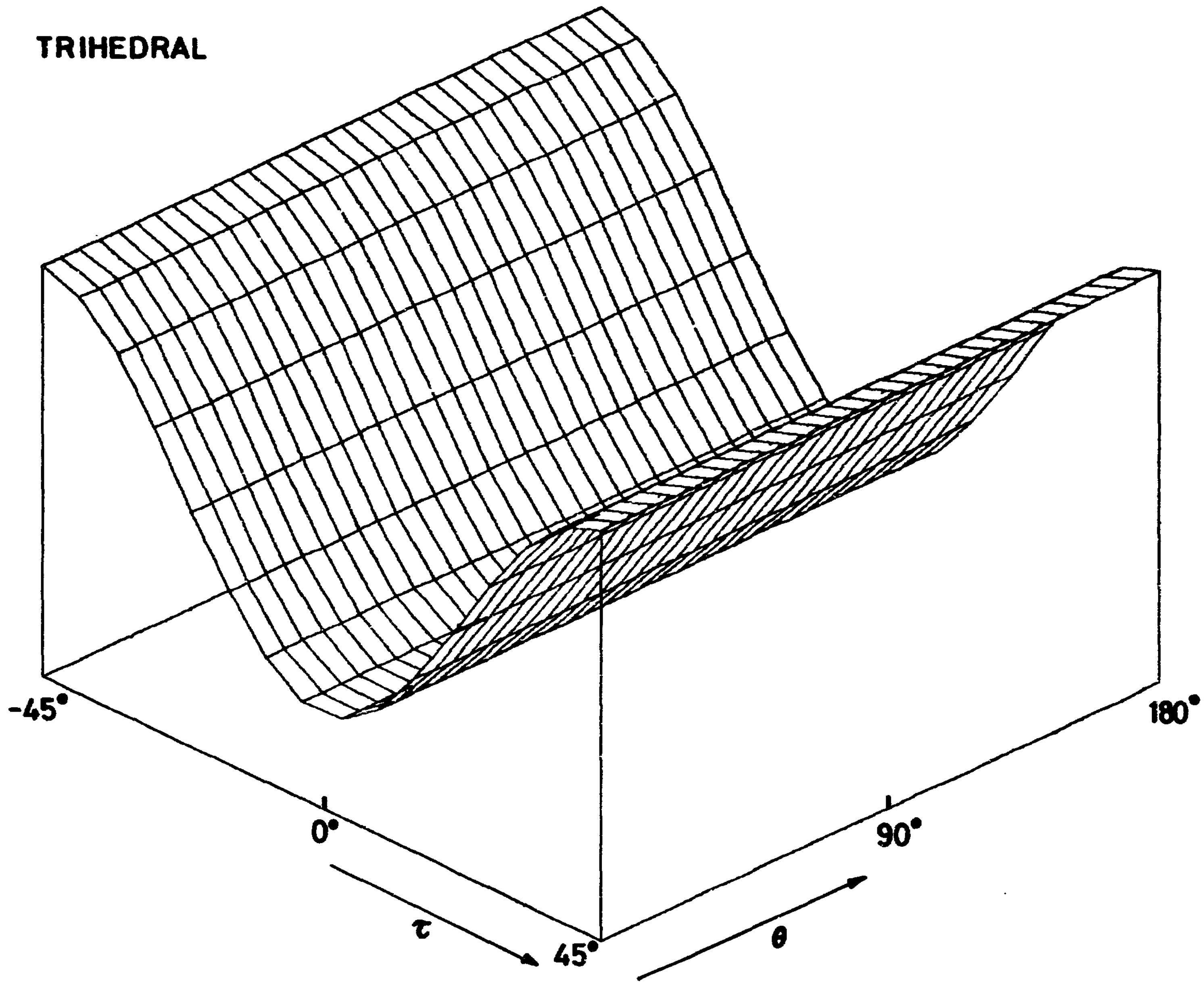
FIG. 19 CROSSPOLAR RCS AS FN OF POLARISATION

TRIHEDRAL



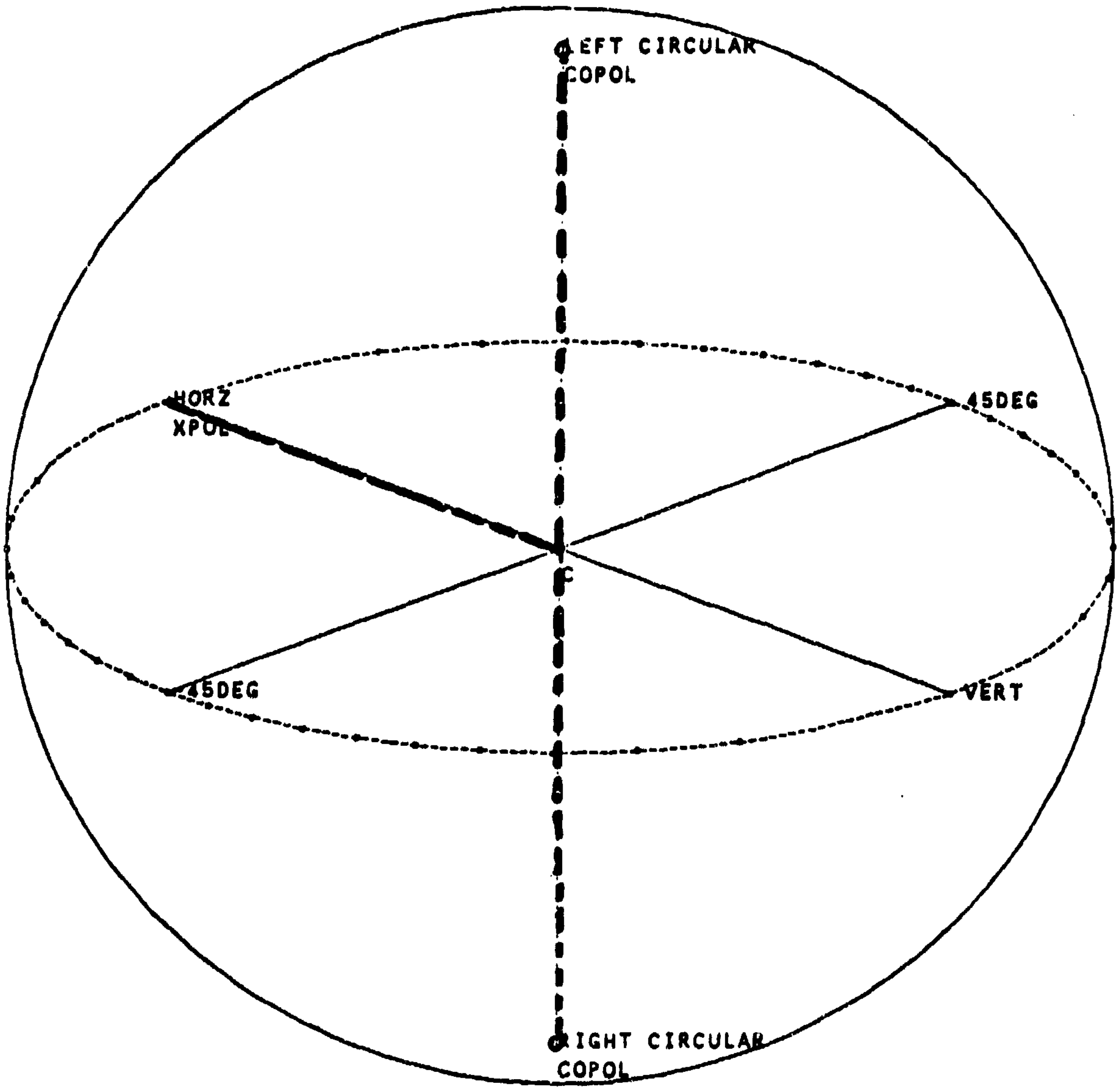


Figure 20: Projection of Poincare Sphere for a Trihedral

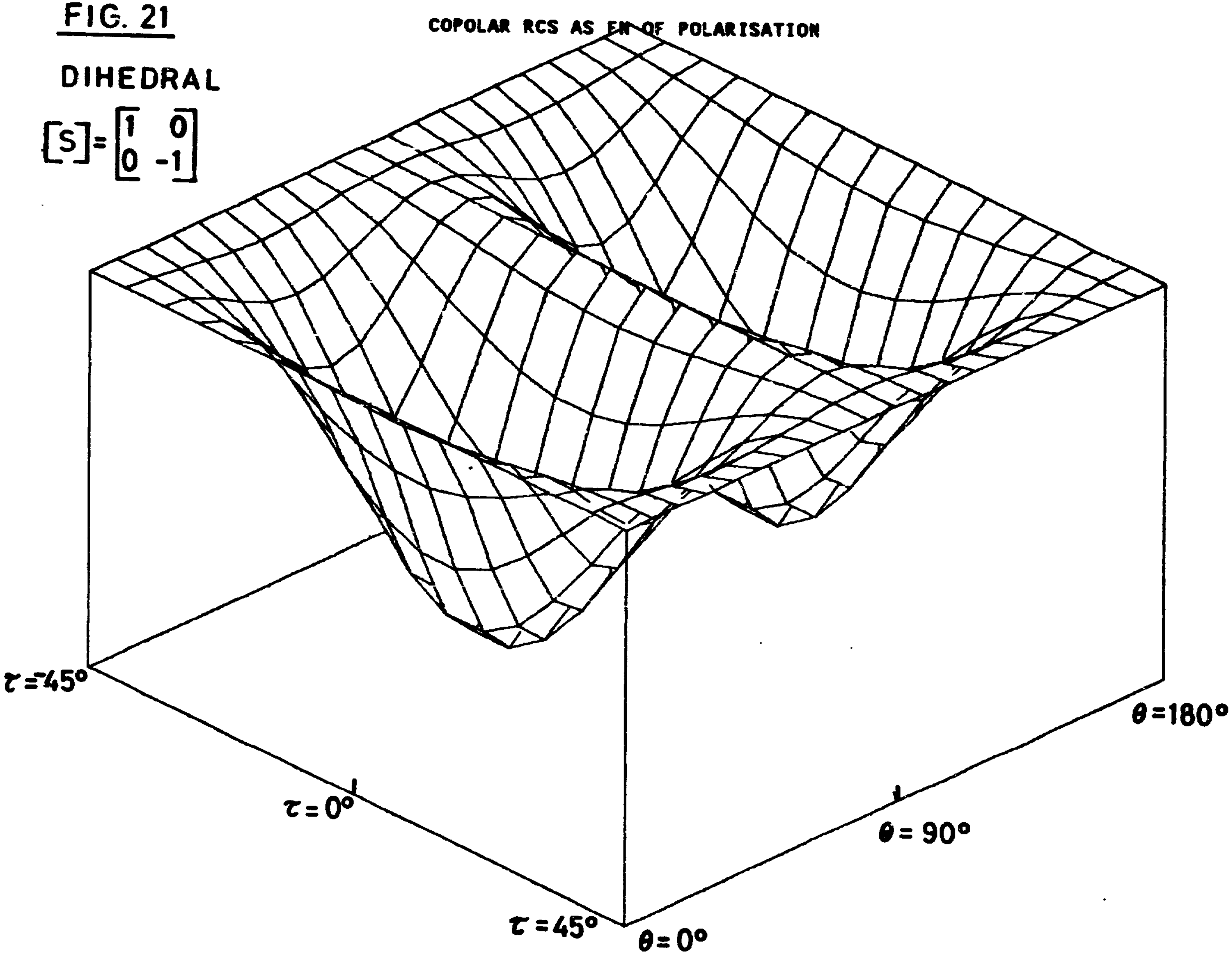


FIG. 21

DIHEDRAL

$$[S]=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

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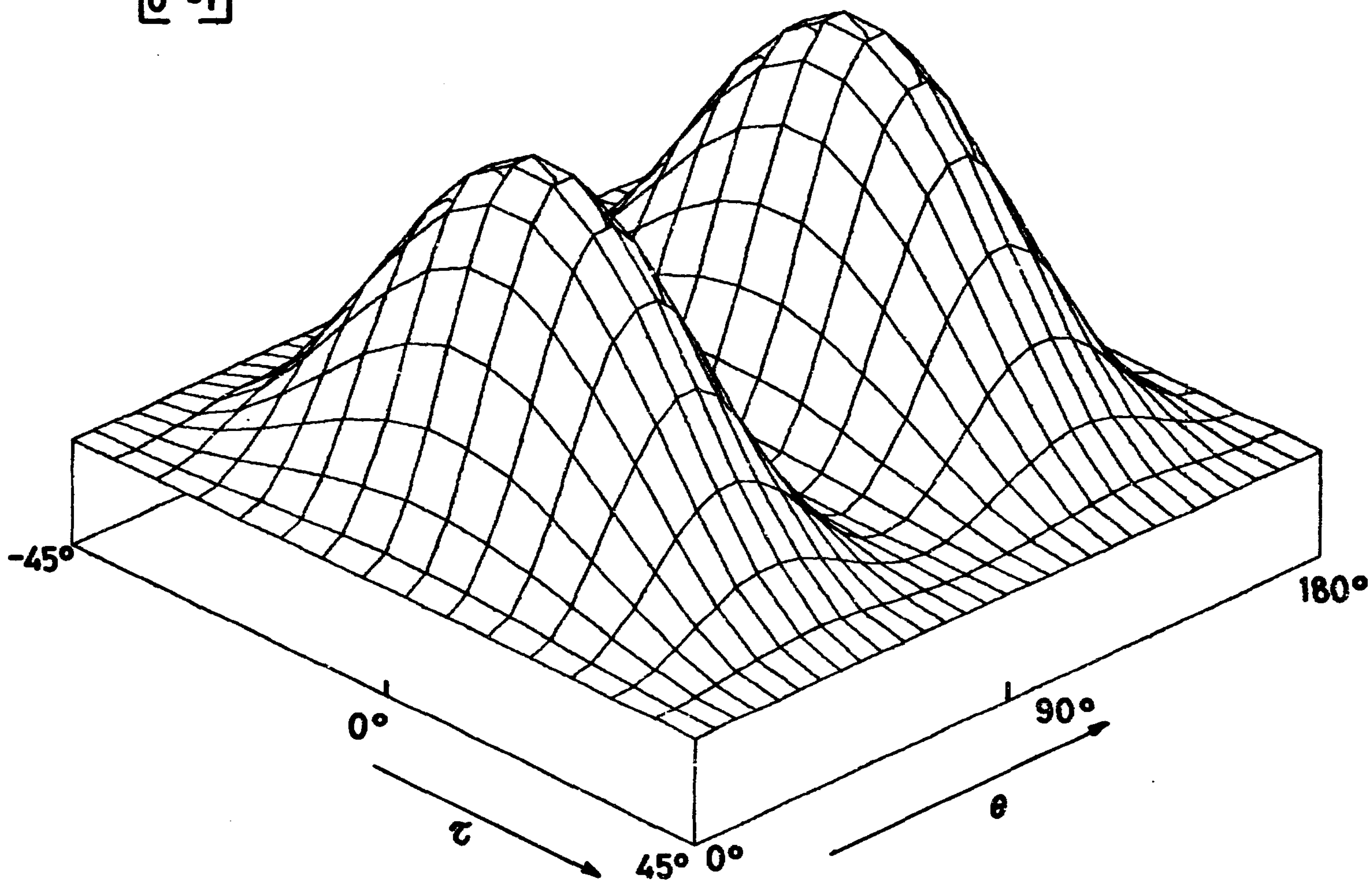
FIG. 22

CROSSPOLAR RCS AS FN OF POLARISATION

DIHEDRAL

$$[S] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



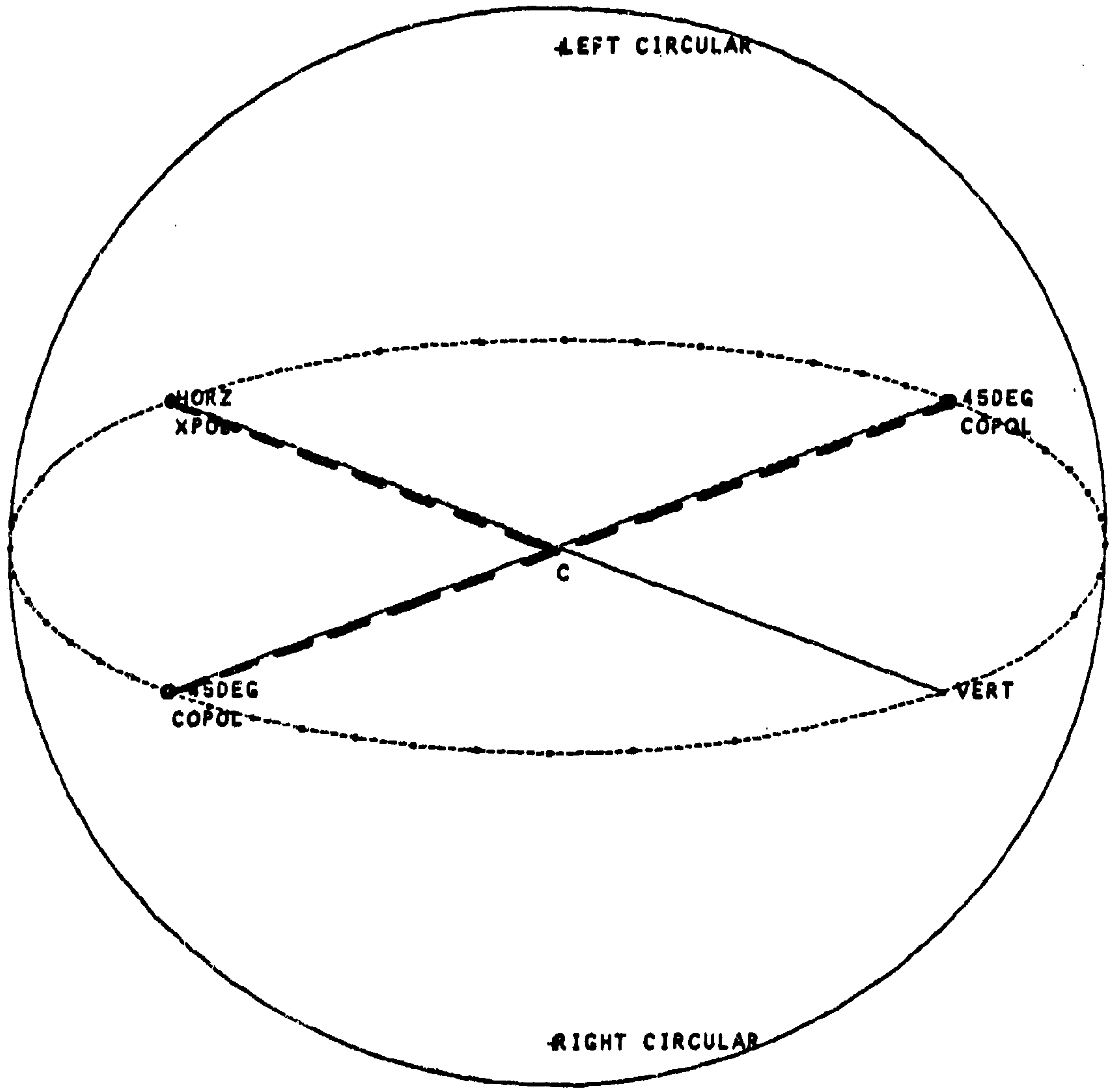


Figure 23: Projection of Poincare Sphere for Dihedral Target

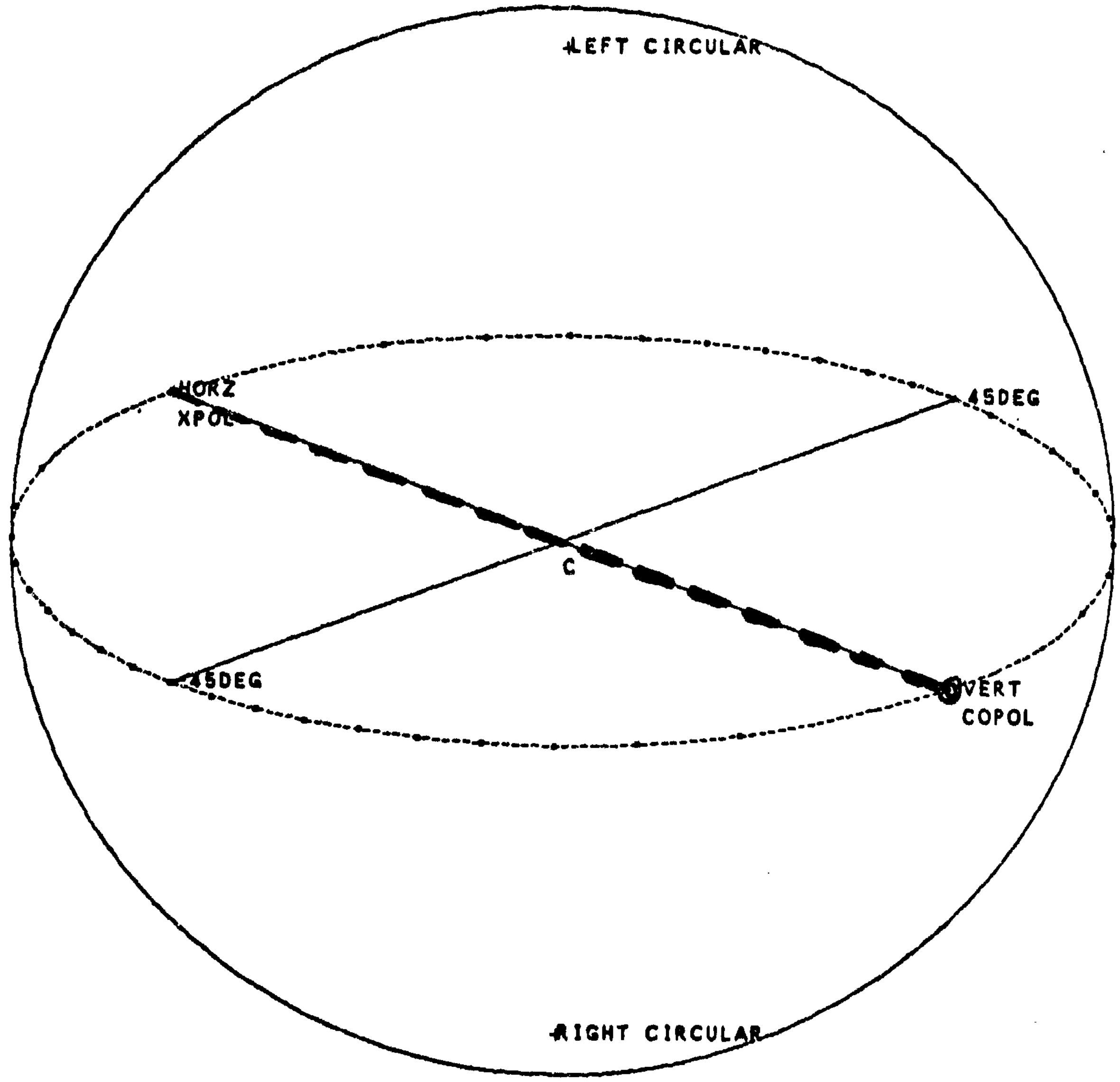


Figure 24: Projection of Poincare Sphere for Linear Target

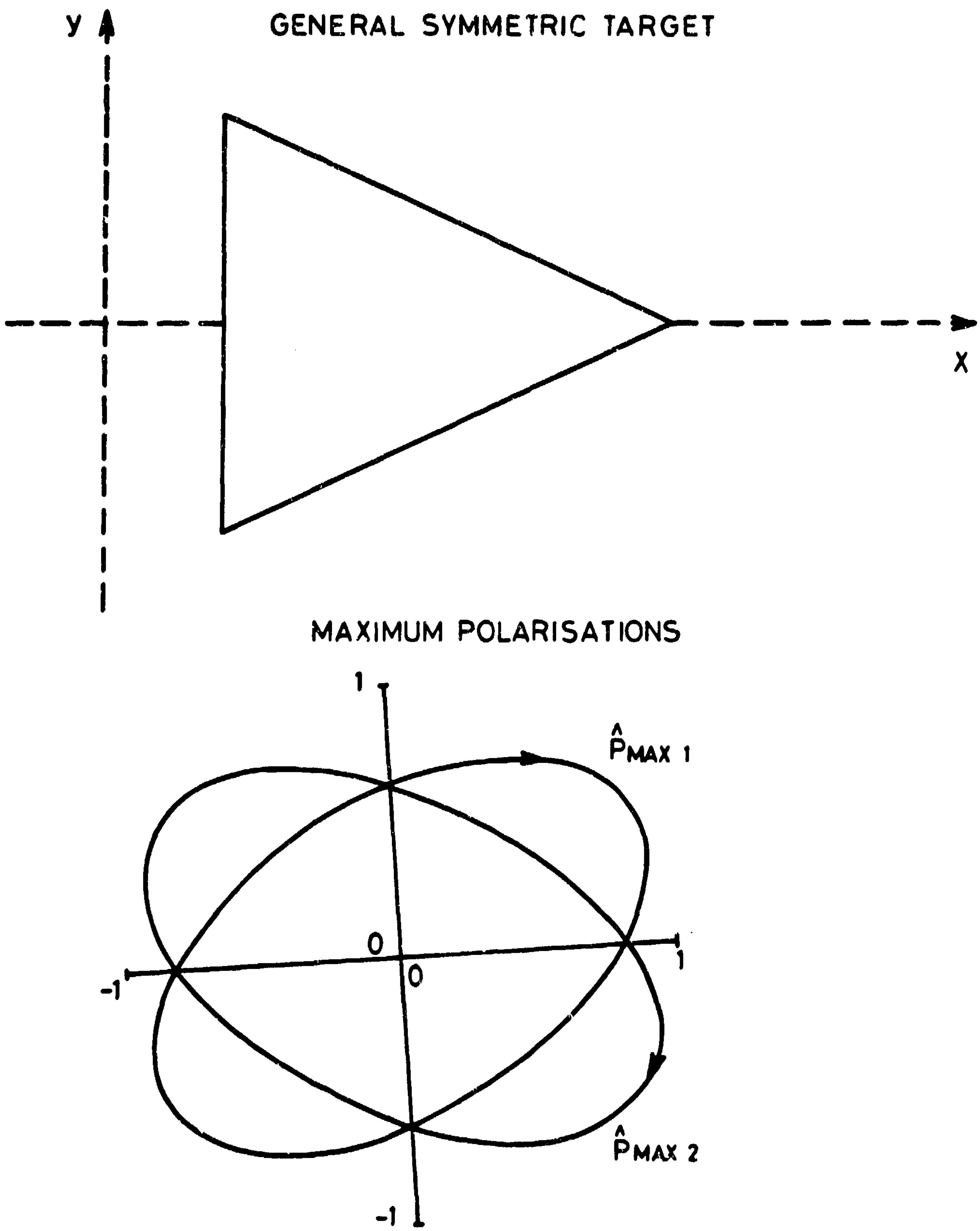

GENERAL SYMMETRIC TARGET
y
X
MAXIMUM POLARISATIONS
1
$\hat{P}_{MAX\ 1}$
0
0
-1
1
$\hat{P}_{MAX\ 2}$
-1

FIG. 25

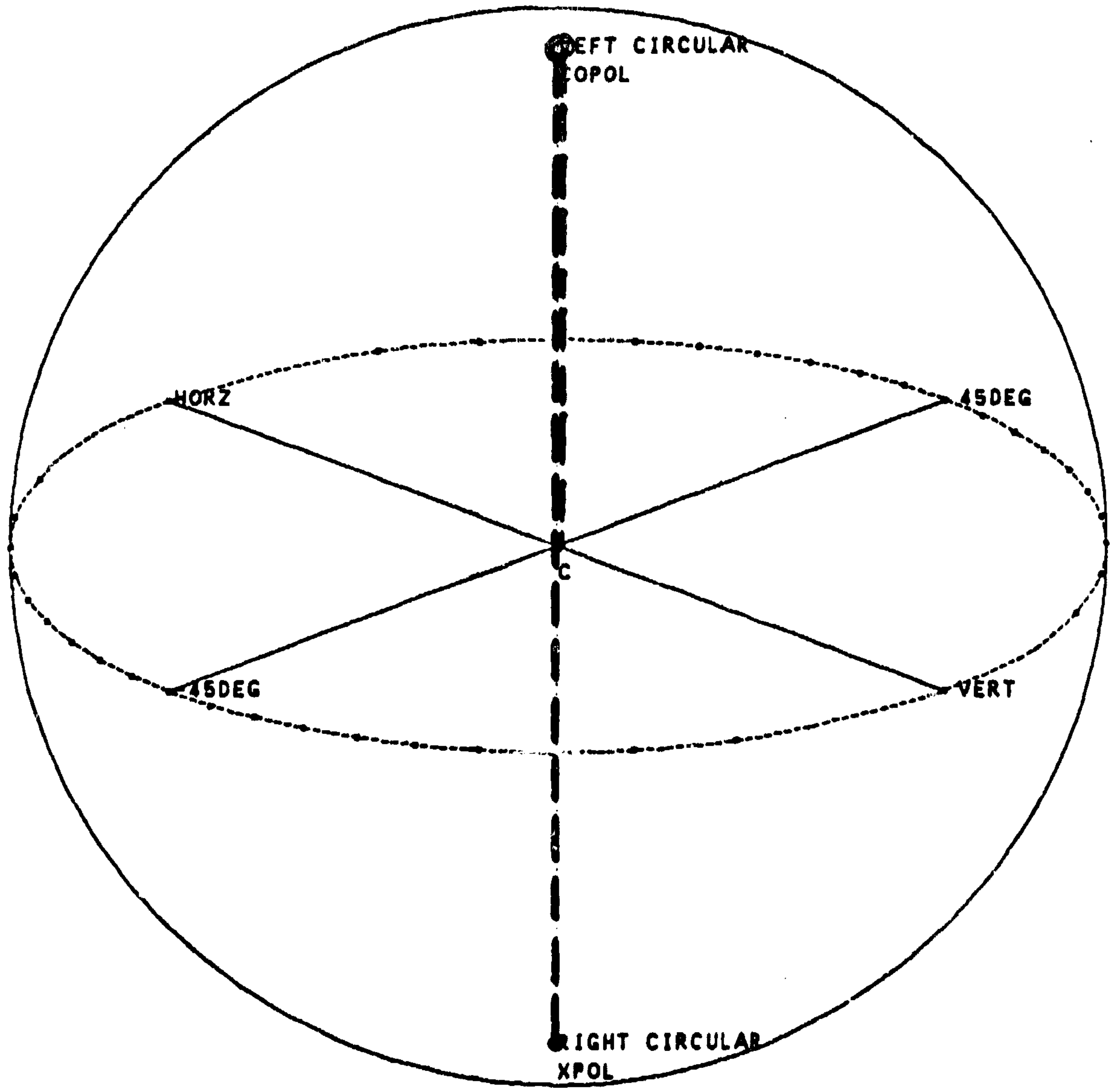


Figure 26: Projection of Poincare Sphere for Left Helix

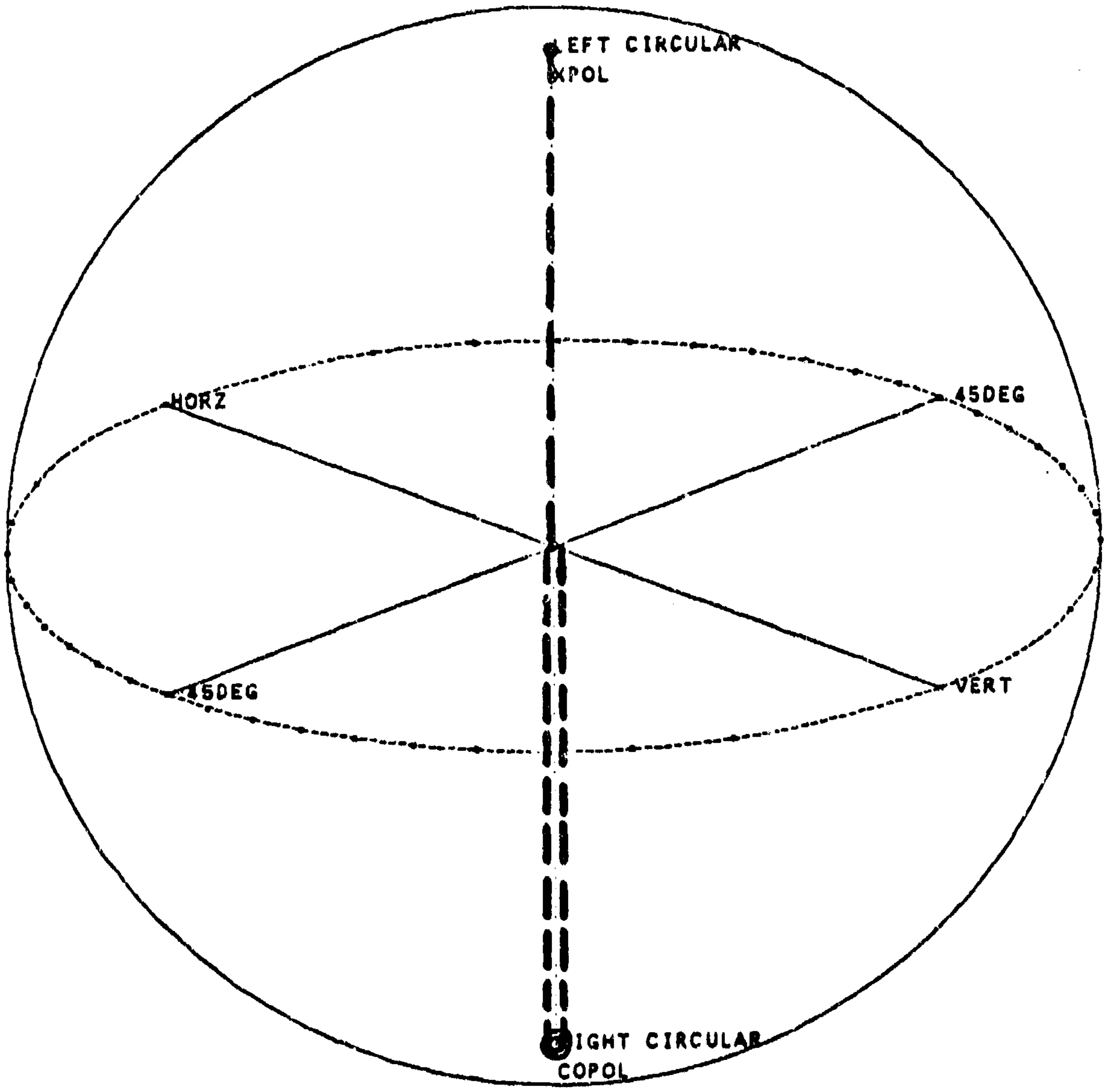


Figure 27: Projection of Poincare Sphere for Right Helix

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	3615		Unclassified

5. Originator's Code (if known)	6. Originator (Corporate Author) Name and Location
	S R Cloude, R011 Royal Signals and Radar Establishment
5a. Sponsoring Agency's Code (if known)	6a. Sponsoring Agency (Contract Authority) Name and Location

7. Title

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7a. Title in Foreign Language (in the case of translations)

7b. Presented at (for conference papers) Title, place and date of conference

8. Author 1 Surname, initials	9(a) Author 2	9(b) Authors 3,4...	10. Date	pp. ref.
Cloude S R				

11. Contract Number	12. Period	13. Project	14. Other Reference
		625	

15. Distribution statement

Descriptors (or keywords)

Polarisation, Radar, Scattering Matrix Theory

continue on separate piece of paper

Abstract

This memo outlines the mathematical formulation of a polarimetric theory for radar scattering. The emphasis is placed on physical interpretation of some fundamental results from the theory of nonsingular linear transformations and the general scattering problem treated as a geometrical transformation on the Poincaré sphere.

An introduction to second order statistical effects in polarimetric scattering is also provided via the coherency matrix and Stokes vectors. The matrix governing transformation of these second order parameters is related to the elements of the coherent scattering matrix.

This memo was derived from a set of lectures given by the author at Birmingham University in July 1983.